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Tracking with prescribed transient behaviour *

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Abstract

Universal tracking control is investigated in the context of a class $\mathcal{S}$ of M-input, M-output dynamical systems modelled by functional differential equations. The class of systems encompasses a wide variety of nonlinear and infinite-dimensional systems and contains – as a prototype subclass – all finite-dimensional linear single-input single-output minimum-phase systems with positive high-frequency gain. The control objective is to ensure that, for an arbitrary $\mathbb{R}^M$-valued reference signal r of class $W^{1,\infty}$ (absolutely continuous and bounded with essentially bounded derivative) and every system of class $\mathcal{S}$, the tracking error e between plant output and reference signal evolves within a prespecified performance envelope or funnel in the sense that $\varphi(t)\|e(t)\| < 1$ for all $t \geq 0$, where φ a prescribed real-valued function of class $W^{1,\infty}$ with the property that $\varphi(s) > 0$ for all $s > 0$ and $\liminf_{s\to\infty} \varphi(s) > 0$. A simple (neither adaptive nor dynamic) error feedback control of the form $u(t) = -\alpha(\varphi(t)\|e(t)\|)e(t)$ is introduced which achieves the objective whilst maintaining boundedness of the control and of the scalar gain $\alpha(\varphi(\cdot)\|e(\cdot)\|)$.

Keywords: nonlinear systems, functional differential equations, feedback control, tracking, transient behaviour.

AMS subject classifications: 93D15, 93C30, 34K20.

1 Introduction

In 1991, Miller and Davison [4] posed the problem "of forcing this error [between plant output and reference signal] to be less than an (arbitrary small) prespecified constant after an (arbitrarily short) prespecified period of time, with an (arbitrarily small) prespecified upper bound on the amount of overshoot." They solved this problem for the rather general class of minimum phase, "stabilizable and detectable, single-input single-output linear time-invariant plant[s]". Their adaptive controller "consists of an LTI compensator together with a switching mechanism to adjust the compensator parameters".

In the present paper, we address a similar problem but with two basic issues which distinguish our formulation from that of [4]: (a) we restrict attention to systems of relative degree one which satisfy a (generalized) positive high-frequency gain condition; (b) we encompass a wide variety of nonlinear infinite-dimensional systems modelled by functional differential equations. When compared with [4], (a) is a severe restriction (when viewed in the linear systems context of [4]) which, however, is counterbalanced by (b), the generality and diversity of nonlinear and

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infinite-dimensional effects allowed in the current paper.
More specifically, our *class of systems* consists of infinite-dimensional, nonlinear M-input u, M-output y systems (p, f, T), given by a controlled nonlinear functional differential equation of the form

$$\dot{y}(t) = f(p(t), (Ty)(t), u(t)), \quad y|_{[-h,0]} = y^0 \in C([-h, 0]; \mathbb{R}^M) \tag{1}$$

where, loosely speaking, $h \geq 0$ quantifies the "memory" of the system, p may be thought of as a (bounded) disturbance term and T is a nonlinear causal operator. Whilst a full description of the system class $\mathcal{S}$ is postponed to Definition 3, we remark here that diverse phenomena are incorporated within the class including, for example, diffusion processes, delays (both point and distributed) and hysteretic effects. We also remark that the class $\mathcal{S}$ is closely related to that of [2]; however, in [2], knowledge of bounding functions relating to f and T is required in order to design an adaptive controller which ensures tracking with prescribed asymptotic accuracy (but which cannot ensure prescribed transient behaviour).
The class $\mathcal{R}$ of *reference signals* is the same as in [4], namely $W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$ (locally absolutely continuous and bounded functions with essentially bounded derivative).

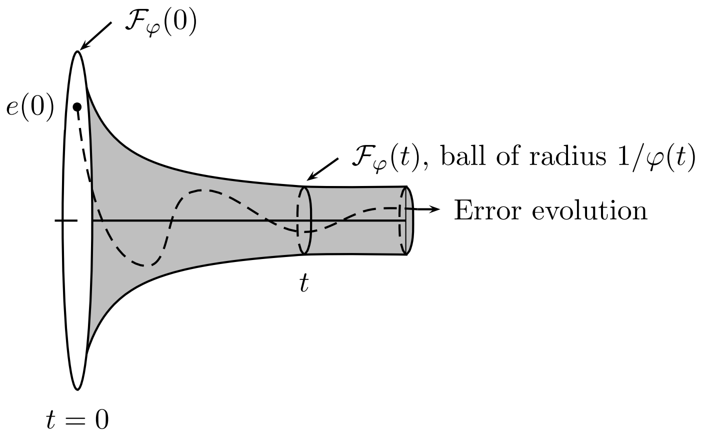


Figure 1: Prescribed performance funnel $\mathcal{F}_\varphi$.

We formulate the control problem in terms of a *performance funnel* $\mathcal{F}_\varphi$, where $\varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$ is a prescribed function with $\varphi(s) > 0$ for all $s > 0$ and $\liminf_{s\to\infty} \varphi(s) > 0$. The reciprocal of φ determines the radius of the funnel:

$$\mathcal{F}_\varphi : \; t \mapsto \{e \in \mathbb{R}^M | \; \varphi(t)\|e\| < 1\},$$

the funnel itself being identified with the graph of the above set-valued map (see Figure 1).
The *objective* is an $(\mathcal{R}, \mathcal{S})$-universal feedback control which, when applied to any system of the admissible class $\mathcal{S}$ with any reference signal of class $\mathcal{R}$, ensures that the tracking error e between plant output and reference signal evolves within the performance funnel $\mathcal{F}_\varphi$ provided that the initial data is such that $e(0) = y^0(0) - r(0) \in \mathcal{F}_\varphi(0)$ (the latter condition is vacuous if $\varphi(0) = 0$).
The *main result* of the paper is that the tracking objective is achieved by a simple (neither adaptive, nor dynamic) time-varying error feedback of the form

$$u(t) = -\alpha\big(\varphi(t)\|e(t)\|\big)\, e(t), \qquad e(t) = y(t) - r(t), \tag{2}$$

where $\alpha : [0, 1) \to \mathbb{R}_{\geq 0}$ is any continuous, unbounded injection: for example, $\alpha(s) = 1/(1 - s)$, in which case the control takes a strikingly simple form $u(t) = -[1 - \varphi(t)\|e(t\|]^{-1}e(t)$.
Underlying the "funnel controller" (2) is the simple idea that, if $e(t)$ approaches the funnel

boundary, then the gain $\alpha(\varphi(t)\|e(t)\|)$ increases: this feature, in conjunction with a high-gain property of the underlying system class, precludes boundary contact. Moreover, in all cases, the gain (and the control) remains bounded and $\|e(\cdot)\|$ is bounded away from the funnel boundary.

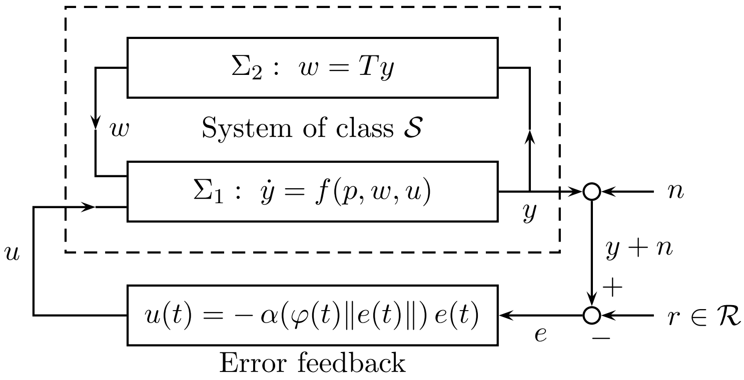


Figure 2: $(\mathcal{R}, \mathcal{S})$-universal error feedback control

The proposed controller (2) also tolerates *output measurement disturbance* n, provided that the disturbance belongs to the same function class as the reference signals. From a strictly analytical viewpoint, in the presence of output disturbances of class $W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$, the disturbance-free analysis is immediately applicable on replacing the reference signal r by the signal $r - n$. Even though the reference signal r and disturbance signal n are assumed to be of the same class, practically, these signals might be distinguished by their respective spectra (n typically having "high-frequency" content). Moreover, from a practical viewpoint, one might reasonably expect that the disturbance n is "small"; if an *a priori* bound on the magnitude of the disturbance is available, then the asymptotic radius of the funnel should be chosen to be commensurate with that bound.

We close this introduction with some remarks on notation.

$\mathbb{R}_{\geq 0}$	$:= [0, \infty)$
$\mathbb{C}_+$	the open right half complex plane
$\mathbb{C}_-$	the open left half complex plane
$\|x\|$	$:= \sqrt{\langle x, x \rangle}$, the Euclidean norm $x \in \mathbb{R}^N$
$\mathbb{B}_r^N(x)$	$:= \{y \in \mathbb{R}^N \mid \|x - y\| < 1\}$, open ball of radius $r > 0$ centred at $x \in \mathbb{R}^N$
$\mathbb{B}_r^N$	$:= \mathbb{B}_r^N(0)$
$\overline{A}$	closure of $A \subset \mathbb{R}^N$
$C(I; \mathbb{R}^N)$	set of continuous functions $I \to \mathbb{R}^N$, $I \subset \mathbb{R}$ an interval
$AC_{\mathrm{loc}}(I; \mathbb{R}^N)$	set of locally absolutely continuous functions $I \to \mathbb{R}^N$, $I \subset \mathbb{R}$ an interval
$L^\infty(I; \mathbb{R}^N)$	space of measurable essentially bounded functions $I \to \mathbb{R}^N$ with norm
$\|x\|_\infty$	$:= \text{ess-sup}_{t \in I} \|x(t)\|$
$L^\infty_{\mathrm{loc}}(I; \mathbb{R}^N)$	space of measurable, locally essentially bounded functions $I \to \mathbb{R}^N$
$W^{1,\infty}(I; \mathbb{R}^N)$	space of bounded functions $x \in AC_{\mathrm{loc}}(I; \mathbb{R}^N)$ with derivative $\dot{x} \in L^\infty(I; \mathbb{R}^N)$ and norm
$\|x\|_{1,\infty}$	$:= \|x\|_\infty + \|\dot{x}\|_\infty$
$x\|_J$	restriction of $x : I \to \mathbb{R}^N$ to $J \subset I$.

2 The class of systems

Here, we make precise the underlying class of systems of form (1), characterized by a triple (p, f, T). We first define the class of operators T.

Definition 1: Operator class $\mathcal{T}$
An operator T is said to be of class $\mathcal{T}$ if, and only if, for some $h \geq 0$ and $N, Q \in \mathbb{N}$, the following hold:

1. $T : C([-h, \infty); \mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^Q)$.
2. *For every $\delta > 0$, there exists $\Delta > 0$ such that, for all $x \in C([-h, \infty); \mathbb{R}^N)$,*
$$\sup_{t \in [-h,\infty)} \|x(t)\| \leq \delta \quad \Longrightarrow \quad \|(Tx)(t)\| \leq \Delta \quad \textit{for almost all } t \geq 0 \,.$$
3. *For all $t \in \mathbb{R}_+$, the following hold:*
   (a) *for all $x, \xi \in C([-h, \infty); \mathbb{R}^N)$,*
   $$x(\cdot) \equiv \xi(\cdot) \ \textit{on } [-h, t] \quad \Longrightarrow \quad (Tx)(s) = (T\xi)(s) \ \ \textit{for almost all } s \in [0, t];$$
   (b) *for all continuous $\zeta : [-h, t] \to \mathbb{R}^N$, there exist $\tau, \delta, c > 0$ such that, for all $x, \xi \in C([-h, \infty); \mathbb{R}^N)$ with $x|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ and $x(s), \xi(s) \in \mathbb{B}_\delta(\zeta(t))$ for all $s \in [t, t + \tau]$,*
   $$\text{ess-sup}_{s\in[t,t+\tau]} \|(Tx)(s) - (T\xi)(s)\| \ \leq \ c \sup_{s\in[t,t+\tau]} \|x(s) - \xi(s)\| \,.$$

Remarks 2

(i) Property 3a is an assumption of causality.

(ii) Property 3b is a technical assumption on T of a "locally Lipschitz" nature.

(iii) Let $T \in \mathcal{T}$ and $t \geq 0$. Given $x \in C([-h, t); \mathbb{R}^N)$ let x^e denote an arbitrary extension of x to $C([-h, \infty); \mathbb{R}^N)$. By virtue of Property 3a, $Tx^e|_{[0,t)}$ is uniquely determined by the function x in the sense that, the former is independent of the extension x^e chosen for the latter. Expanding on this observation, we will adopt the following notational convention. For $s \in [0, t)$, we simply write $(Tx)(s)$ in place of $(Tx^e)(s)$ (where $x^e \in C([-h, \infty); \mathbb{R}^N)$ is any continuous extension of x).

(iv) For each $\omega \in \mathbb{R}$, let S_ω denote the shift operator given by $(S_\omega x)(t) := x(t + \omega)$ for all $t \in \mathbb{R}$. Then
$$T \in \mathcal{T} \quad \Longrightarrow \quad S_\omega T S_{-\omega} \in \mathcal{T} \quad \text{for all } \ \omega \geq 0 \,. \tag{3}$$

Now, we define the class of systems underlying the paper.

Definition 3: System class $\mathcal{S}$

$\mathcal{S}$ is the class of nonlinear M-input u, M-output y systems (p, f, T), given by a controlled nonlinear functional differential equation of the form (1), where $h \geq 0$ quantifies the "memory" of the system and, for some $P, Q \in \mathbb{N}$,

1. $p \in L^\infty(\mathbb{R}; \mathbb{R}^P)$,

2. $f:\mathbb{R}^P\times\mathbb{R}^Q\times\mathbb{R}^M\to\mathbb{R}^M$ *is continuous,*
3. *for every non-empty compact set* $\mathcal{C}\subset\mathbb{R}^P\times\mathbb{R}^Q$ *and sequence* $(u_n)\subset\mathbb{R}^M\setminus\{0\}$, *the following property (akin to radial unboundedness or weak coercivity) holds:*

$$\|u_n\|\to\infty \quad as \quad n\to\infty \qquad \Longrightarrow \qquad \min_{(v,w)\in\mathcal{C}}\frac{\langle u_n, f(v,w,u_n)\rangle}{\|u_n\|}\to\infty \quad as \quad n\to\infty\,. \tag{4}$$

4. $T: C([-h,\infty);\mathbb{R}^M)\to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{R}^Q)$ *is of class* $\mathcal{T}$.

Remarks 4

(i) Property 3 of Definition 3 generalizes the positive "high-frequency gain" concept in linear systems, as will be discussed in Sub-section 4.1.

(ii) It is straightforward to show that a necessary and sufficient condition for Property 3 of Definition 3 to hold is that, for $\mathbb{S}^{M-1}:=\{u\in\mathbb{R}^M|\ \|u\|=1\}$ and for every compact set $\mathcal{C}\subset\mathbb{R}^P\times\mathbb{R}^Q$, the continuous function $\gamma_\mathcal{C}:\mathbb{R}_{\ge 0}\to\mathbb{R}$, defined below, has the following property:

$$\min_{(u,v,w)\in\mathbb{S}^{M-1}\times\mathcal{C}}\langle u, f(v,w,su)\rangle =: \gamma_\mathcal{C}(s)\to\infty \qquad \text{as} \qquad s\to\infty. \tag{5}$$

(iii) An important consequence of Property 3 and Remark (ii) above is that, for every non-empty compact set $\mathcal{C}\subset\mathbb{R}^P\times\mathbb{R}^Q$,

$$\langle e,\ f(v,w,-ke)\rangle \ \le\ -\gamma_\mathcal{C}(k\|e\|)\,\|e\| \qquad \text{for all } (v,w,e,k)\in\mathcal{C}\times\mathbb{R}^M\times\mathbb{R}\,. \tag{6}$$

This anticipates the rôle of $\gamma_\mathcal{C}$ in Lypunov-type analyses in later proofs.

(iv) Suppose that (p,f,T) satisfies Properties 1, 2, 4 of Definition 3, but instead of Property 3 we have,

3a. there exists known, *symmetric, positive-definite* $G\in\mathbb{R}^{M\times M}$ *such that, for every non-empty compact set* $\mathcal{C}\subset\mathbb{R}^P\times\mathbb{R}^Q$ *and sequence* $(u_n)\subset\mathbb{R}^M\setminus\{0\}$, *the following property holds:*

$$\|u_n\|_G\to\infty \text{ as } n\to\infty \qquad \Longrightarrow \qquad \min_{(v,w)\in\mathcal{C}}\frac{\langle u_n, Gf(v,w,u_n)\rangle}{\|u_n\|_G}\to\infty \text{ as } n\to\infty\,, \tag{7}$$

where $\|u\|_G:=\|G^{\frac12}u\|$ *for all* $u\in\mathbb{R}^M$.

We show that, no additional generality results if Property 3 is replaced by Property 3a (in which case, one simply replaces the norm $\|\cdot\|$ in (2) by the "G-induced" norm $\|\cdot\|_G$.) Defining $\hat f:(v,w,\hat u)\mapsto \hat f(v,w,\hat u):=G^{\frac12}f(v,w,G^{-\frac12}\hat u)$, we have

$$\frac{\langle \hat u, \hat f(v,w,\hat u)\rangle}{\|\hat u\|}=\frac{\langle u, Gf(v,w,u)\rangle}{\|u\|_G} \quad \text{for all } (\hat u,v,w)=\big(G^{\frac12}u,v,w\big)\in\mathcal{C}\times\mathbb{R}^M.$$

Therefore, (7) implies that $\hat f$ has Property 3 of Definition 3. Defining $\hat T\in\mathcal{T}$ by $\hat T:=TG^{-\frac12}$, it follows that $(p,\hat f,\hat T)$ is of class $\mathcal{S}$. Under the coordinate transformations $\hat y=G^{\frac12}y$ and $\hat u=G^{\frac12}u$, (1) is equivalent to

$$\dot{\hat y}(t)=\hat f(p(t),(\hat T\hat y)(t),\hat u(t)), \quad \hat y|_{[-h,0]}=\hat y^0\in C([-h,0];\mathbb{R}^M)\,. \tag{8}$$

Hence, application of the control $\hat u(t)=-\alpha(\varphi(t)\|\hat e(t)\|)\hat e(t)$ to (8), with reference signal $\hat r=G^{\frac12}r\in\mathcal{R}$, is equivalent to applying $u(t)=-\alpha\big(\varphi(t)\|e(t)\|_G\big)e(t)$ to (1) with reference signal $r\in\mathcal{R}$.

3 Control objectives

The overall control objective is twofold in nature. The *primary objective* may be summarized as that of tracking with prescribed asymptotic accuracy. Precisely, given $\lambda > 0$, a control strategy is sought which, for each system (p, f, T) of class $\mathcal{S}$ and every reference signal $r \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$, when applied to (1) achieves the following: for all (admissible) initial data, the initial-value problem for the closed-loop system has a solution, every solution can be maximally extended, every maximal solution is forward complete, bounded and such that $\|e(t)\| < \lambda$ for all t sufficiently large, where $e(t) := y(t) - r(t)$ denotes the tracking error. The *secondary objective* pertains to transient behaviour: in addition to achieving the primary objective, the control should shape the error transient in the sense that the evolution of the tracking error is required to satisfy prescribed constraints.

We capture both objectives by the requirement that, under feedback control, the tracking error e should be such that $\varphi(t)\|e(t)\| < 1$ for all $t \geq 0$, where $\varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$ is a prescribed function with $\varphi(s) > 0$ for all $s > 0$. For example, for $\lambda > 0$, $\tau > 0$ and $\varepsilon \in (0, 1)$, the choice

$$t \mapsto \varphi(t) = \frac{t}{([1-\varepsilon]t + \varepsilon\tau)\lambda} \tag{9}$$

corresponds to an overall objective of attaining prescribed tracking accuracy $\lambda > 0$ in prescribed time $\tau > 0$.

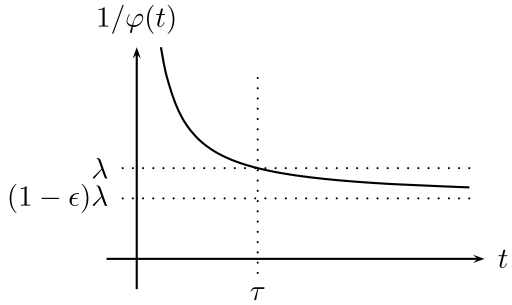


Figure 3: Typical funnel radius.

In summary and with reference to Figures 1 and 3, the control objective can be viewed in terms of a performance funnel

$$\mathcal{F}_\varphi : \ t \mapsto \{e \in \mathbb{R}^M \mid \varphi(t)\|e\| < 1\},$$

the radius of which is determined by the reciprocal of a prescribed function $\varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$ with $\varphi(s) > 0$ for all $s > 0$: a control is sought which, when applied to any system of class $\mathcal{S}$ with any reference signal r of class $W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$ and with initial data satisfying $\varphi(0)\|y^0(0) - r(0)\| < 1$, ensures that the tracking error $e = y - r$ evolves within the funnel $\mathcal{F}_\varphi$.

Reiterating remarks made in the Introduction, the main result of the paper is that the control objective is achieved by a feedback strategy of form (2), where $\alpha : [0, 1) \to \mathbb{R}_{\geq 0}$ is any continuous, unbounded injection.

Note that:

(a) if $\varphi(0) = 0$, then the constraint on the initial data is vacuous and so the results are global;

(b) if $\varphi(0) > 0$, then the initial data has to satisfy $\|e(0)\| < 1/\varphi(0)$ and so the results are of a semi-global nature. Such a semi-global control may arise from a requirement for maximal *error overshoot*. For example, if for some $\delta > 0$ one requires $\|e(t)\| < \|e(0)\| + \delta$ for all $t \geq 0$, then one has to choose φ of the admissible class with $(1/\varphi(0)) \in (\|e(0)\|, \|e(0)\| + \delta]$ and $(1/\varphi(t)) \leq \|e(0)\| + \delta$ for all $t \geq 0$. Then the feedback control will ensure that $\|e(t)\| < \|e(0)\| + \delta$ for all $t \geq 0$.

4 Sub-classes of $\mathcal{S}$

Here, we highlight some particular sub-classes of the general system class $\mathcal{S}$.

4.1 The finite-dimensional linear prototype

Consider the prototype class of finite-dimensional, real, linear, minimum-phase, M-input $(u(t))$, M-output $(y(t))$ systems of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\ x(0) = x^0, \\ y(t) &= Cx(t),\end{aligned} \qquad \det\begin{bmatrix} sI_n - A & B \\ C & 0\end{bmatrix} \neq 0\ \forall s \in \overline{\mathbb{C}}_+, \quad (CB)^T + CB > 0\,, \tag{10}$$

with $x(t) \in \mathbb{R}^N$ (N arbitrary) and real matrices A, B, C of conforming formats. In (10), the condition on the determinant characterizes the minimum-phase assumption; the second condition requires the high-frequency gain $CB \in \mathbb{R}^{M\times M}$ to have positive definite symmetric part. Since the latter ensures invertibility of CB, which gives $\mathbb{R}^N = \operatorname{im} B \oplus \ker C$, there exists $V \in \mathbb{R}^{N\times(N-M)}$, with $\operatorname{im} V = \ker C$, such that the coordinate transformation

$$x \mapsto \begin{bmatrix} y \\ z \end{bmatrix} := S^{-1}x \quad \text{where } S := \left[B(CB)^{-1} \,\vdots\, V\right]$$

takes (10) into the equivalent form

$$\left.\begin{aligned}\dot{y}(t) &= A_1 y(t) + A_2 z(t) + CB\,u(t), & y(0) &= y^0, & (CB)^T + CB &> 0, \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t), & z(0) &= z^0, & \sigma(A_4) &\subset \mathbb{C}_-,\end{aligned}\right\} \tag{11}$$

with $z(t) \in \mathbb{R}^{N-M}$, and real matrices A_1, A_2, A_3, A_4 of conforming formats.
By the minimum-phase condition, A_4 has spectrum in the open left half complex plane, and so by setting

$$(Ty)(t) := A_1 y(t) + A_2 \int_0^t \exp(A_4(t-s))A_3 y(s)ds, \quad p(t) := \begin{cases} A_2 \exp(A_4 t)z^0 & t \geq 0 \\ 0 & t < 0\,, \end{cases} \tag{12}$$

the linear operator $T : C(\mathbb{R}_{\geq 0}; \mathbb{R}^M) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$ belongs to $\mathcal{T}$ and p belongs to $L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$. Defining

$$f : \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M,\ (v, w, u) \mapsto v + w + CBu\,,$$

and setting $h = 0$, we may recast system (11) in the form (1):

$$\dot{y}(t) = p(t) + (Ty)(t) + CBu(t), \quad y(0) = y^0 \in \mathbb{R}^M\,. \tag{13}$$

With reference to Figure 2, $y \mapsto T(y)$ and (13), respectively, correspond to components Σ_2 and Σ_1 of the interconnected system.

Finally, noting that, for every non-empty compact set $\mathcal{C} \subset \mathbb{R}^M \times \mathbb{R}^M$,

$$\begin{aligned}\|u\| \max_{(v,w)\in\mathcal{C}} \|v + w\| + \langle u, [(CB)^T + CB]u\rangle &\geq \min_{(v,w)\in\mathcal{C}} \langle u, f(v, w, u)\rangle \\ &\geq -\|u\| \max_{(v,w)\in\mathcal{C}} \|v + w\| + \langle u, [(CB)^T + CB]u\rangle \quad \text{for all } u \in \mathbb{R}^M\,,\end{aligned}$$

we see that Property 3 of Definition 3 is equivalent to the assumption $(CB)^T + CB > 0$.

4.2 Infinite-dimensional regular linear systems

The finite-dimensional class of systems considered in (11) can be extended to an infinite-dimensional setting by reinterpreting the operators A_1, A_2, A_3 and A_4 as the generating operators of a regular linear system (regular in the sense of [8]). In particular, in this setting, A_4 is assumed to be the generator of a strongly continuous semigroup $\mathbf{S} = (\mathbf{S}_t)_{t\geq 0}$ of bounded linear operators on a Hilbert space X with norm $\|\cdot\|_X$. Let X_1 denote the space $\mathrm{dom}(A_4)$ endowed with the graph norm and X_{-1} denotes the completion of X with respect to the norm $\|z\|_{-1} = \|(s_0 I - A_4)^{-1} z\|_X$ where s_0 is any fixed element of the resolvent set of A_4. Then A_3 is assumed to be a bounded linear operator from $\mathbb{R}^M$ to X_{-1} and A_2 is assumed to be a bounded linear operator from X_1 to $\mathbb{R}^M$. $A_1 \in \mathbb{R}^{M\times M}$ is the feedthrough operator of the regular linear system. Finally, $CB \in \mathbb{R}^{M\times M}$ is as in (11).

If we assume that the semigroup $\mathbf{S}$ is exponentially stable and that the operator A_2 extends to a bounded linear operator (again denoted by A_2) from X to $\mathbb{R}^M$, then the operator

$$(Ty)(t) := A_1 y(t) + A_2 \int_0^t \mathbf{S}_{t-s} A_3 y(s)\, ds \tag{14}$$

is of class $\mathcal{T}$ (for details, see [5]) and the overall system can be recast in the form (13).

4.3 Nonlinear systems

Consider the following nonlinear generalization of (11)

$$\left.\begin{aligned} \dot{y}(t) &= Y_1(p(t), y(t), z(t)) + Y_2(y(t), z(t), u(t)), & y(0) &= y^0 \in \mathbb{R}^M \\ \dot{z}(t) &= Z(t, z(t), y(t)), & z(0) &= z^0 \in \mathbb{R}^L\,, \end{aligned}\right\} \tag{15}$$

with continuous $Y_1 : \mathbb{R}^P \times \mathbb{R}^M \times \mathbb{R}^L \to \mathbb{R}^M$, $Y_2 : \mathbb{R}^M \times \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}^M$ and $Z : \mathbb{R}_{\geq 0} \times \mathbb{R}^M \times \mathbb{R}^L \to \mathbb{R}^L$ having the properties: $Z(\cdot, y, z)$ measurable for all $(y, z) \in \mathbb{R}^M \times \mathbb{R}^L$ and, for every compact $\mathcal{C} \subset \mathbb{R}^M \times \mathbb{R}^L$, there exists $\kappa \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ such that $\|Z(t, y, z) - Z(t, \bar{y}, \bar{z})\| \leq \kappa(t)\|(y, z) - (\bar{y}, \bar{z})\|$ for almost all $t \in \mathbb{R}_{\geq 0}$ and all $(y, z), (\bar{y}, \bar{z}) \in \mathcal{C}$.
Then, viewing the second of the differential equations in (15) in isolation (with input y), it follows that, for each $(z^0, y) \in \mathbb{R}^L \times L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$, the initial-value problem $\dot{z}(t) = Z(t, y(t), z(t))$, $z(0) = z^0 \in \mathbb{R}^L$ has unique maximal solution, which we denote by $[0, \omega) \to \mathbb{R}^L$, $t \mapsto z(t; z^0, y)$.
In addition, we assume there exist $c_0 > 0$ and $q > 1$ such that

$$\langle u, Y_2(y, z, u)\rangle \geq c_0\, \|u\|^q \qquad \text{for all} \quad (u, y, z) \in \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^L\,, \tag{16}$$

and there exists a function $\theta \in C(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$ such that, for some constant $c > 0$, and for all $y \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$,

$$\|z(t, z^0, y)\| \leq c\,[1 + \text{ess-sup}_{s\in[0,t]}\theta(\|y(s)\|)] \quad \text{for all } t \in [0, \omega) \tag{17}$$

which, in turn, implies that $\omega = \infty$. Note that this is akin to, but weaker than, Sontag's [6] concept of input-to-state stability.
We show that systems of the class (15) satisfying the above smoothness properties and in particular (16) and (17) belong to the class $\mathcal{S}$. To this end, fix $z^0 \in \mathbb{R}^L$ arbitrarily. Define the operator

$$T : C(\mathbb{R}_{\geq 0}; \mathbb{R}^M) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^M \times \mathbb{R}^L), \quad y \mapsto Ty = (y(\cdot), z(\cdot, z^0, y)).$$

In view of (17), Property 2 of Definition 1 holds; setting $h = 0$, we see that Property 3a of Definition 1 also holds. Arguing as in [5, §3.2.3], via an application of Gronwall's Lemma, it can

be shown that Property 3b holds. Therefore, this construction yields a family (parameterized by the initial data z^0) of operators T of class $\mathcal{T}$.
Defining $f : \mathbb{R}^P \times \mathbb{R}^{M+L} \times \mathbb{R}^M \to \mathbb{R}^M$, $(v, w, u) \mapsto Y_1(v, w) + Y_2(w, u)$ and assuming $p \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^P)$, system (15) may be expressed in the form (1) (with $h = 0$). Let $\mathcal{C} \subset \mathbb{R}^P \times \mathbb{R}^{M \times L}$ be compact. Then, invoking (16),

$$\min_{(v,w)\in\mathcal{C}} \frac{\langle u, f(v, w, u)\rangle}{\|u\|} \geq - \max_{(v,w)\in\mathcal{C}} \|Y_1(v, w)\| + c_0 \|u\|^{q-1} \quad \text{for all } u \in \mathbb{R}^M ,$$

whence Property 3 of Definition 3. Therefore, $(p, f, T) \in \mathcal{S}$.

4.4 Nonlinear delay systems

Let functions $\Psi_i : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^Q : (t, y) \mapsto \Psi_i(t, y)$, $i = 0, ..., n$ be measurable in t and locally Lipschitz in y uniformly with respect to t: precisely, (i) for each fixed y, $\Psi_i(\cdot, y)$ is measurable and (ii) for every compact $\mathcal{C} \subset \mathbb{R}^N$ there exists a constant c such that

$$\text{for almost all } \; t, \quad \|\Psi_i(t, y) - \Psi_i(t, z)\| \leq c \|y - z\| \quad \text{for all } \; y, z \in \mathcal{C} .$$

For $i = 0, ...n$, let $h_i \in \mathbb{R}_{\geq 0}$ and define $h := \max_i h_i$. For $y \in C([-h, \infty); \mathbb{R}^N)$, let

$$(Ty)(t) := \int_{-h_0}^{0} \Psi_0(s, y(t + s))\, ds + \sum_{i=1}^{n} \Psi_i(t, y(t - h_i)) \quad \text{for all } t \geq 0 .$$

The operator T, so defined, is of class $\mathcal{T}$: for details see [5]. Therefore, for $p \in L^\infty(\mathbb{R}; \mathbb{R}^P)$ and continuous $f : \mathbb{R}^P \times \mathbb{R}^Q \times \mathbb{R}^M \to \mathbb{R}^M$ with the Property 3 of Definition 3, (p, f, T) defines an admissible system of class $\mathcal{S}$.

4.5 Systems with hysteresis

A general class of nonlinear operators $C(\mathbb{R}_{\geq 0}; \mathbb{R}) \to C(\mathbb{R}_{\geq 0}; \mathbb{R})$, which includes many physically motivated hysteretic effects, is shown to be of class $\mathcal{T}$ in [2]. See also [3]. Examples of such operators include relay hysteresis, backlash hysteresis, elastic-plastic hysteresis and Preisach operators. Therefore, for any such operator T, and assuming that $p \in L^\infty(\mathbb{R}; \mathbb{R}^P)$ and that $f : \mathbb{R}^P \times \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}^M$ is continuous with Property 3 of Definition 3, (p, f, T) defines a system of class $\mathcal{S}$. Below, we describe two examples of hysteresis operators of class $\mathcal{T}$.

4.5.1 Relay hysteresis

Let $a_1 < a_2$ and let $\rho_1 : [a_1, \infty) \to \mathbb{R}$, $\rho_2 : (-\infty, a_2] \to \mathbb{R}$ be continuous, globally Lipschitz and satisfy $\rho_1(a_1) = \rho_2(a_1)$ and $\rho_1(a_2) = \rho_2(a_2)$. For a given input $y \in C(\mathbb{R}_{\geq 0}; \mathbb{R})$ to the hysteresis element, the output w is such that $(y(t), w(t)) \in \text{graph}(\rho_1) \cup \text{graph}(\rho_2)$ for all $t \in \mathbb{R}_{\geq 0}$: the value $w(t)$ of the output at $t \in \mathbb{R}_{\geq 0}$ is either $\rho_1(y(t))$ or $\rho_2(y(t))$, depending on which of the threshold values a_2 or a_1 was "last" attained by the input y. When suitably initialized, such a hysteresis element has the property that, to each input $y \in C(\mathbb{R}_{\geq 0}; \mathbb{R})$, there corresponds a unique output $w = Ty \in C(\mathbb{R}_{\geq 0}; \mathbb{R})$: the operator T, so defined, is of class $\mathcal{T}$ with $N = Q = 1$. This situation is illustrated by Figure 3.

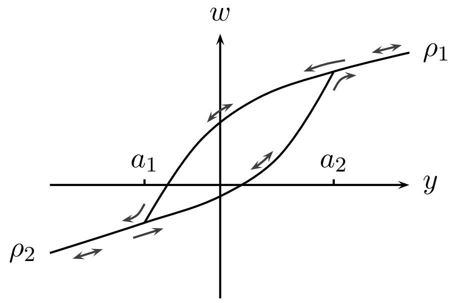
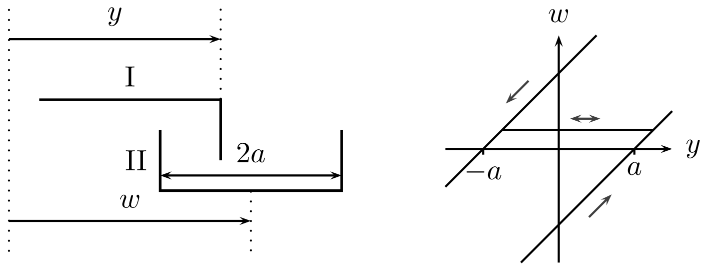


Figure 3: Relay hysteresis

4.5.2 Backlash hysteresis

Consider a one-dimensional mechanical link consisting of the two solid parts I and II, as shown in Figure 4a, the displacements of which (with respect to some fixed datum) at time $t \geq 0$ are given by $y(t)$ and $w(t)$ with $|y(t) - w(t)| \leq a$ for all t, and $w(0) := y(0) + \xi$ for some pre-specified $-a \leq \xi \leq a$. Within the link there is mechanical play: that is to say the position $w(t)$ of II remains constant as long as the position $y(t)$ of I remains within the interior of II. Thus, assuming continuity of y, we have $\dot{w}(t) = 0$ whenever $|y(t)-w(t)| < a$. Given a continuous input $y \in C(\mathbb{R}_{\geq 0}; \mathbb{R})$, describing the evolution of the position of I, denote the corresponding position of II by $w = Ty$. The operator T, (in effect we define a family T_ξ of operators parameterized by the initial offset ξ) so defined, is known as *backlash* or *play* and is of class $\mathcal{T}$ with $N = Q = 1$.



(a) (b)

Figure 4: Backlash hysteresis

5 An existence theorem

The potential singularity in the proposed control (2) raises a basic question: is the resulting closed-loop initial-value problem well posed? In due course, we will answer this question affirmatively. To this end, we first provide an existence theory for initial-value problems of a form of sufficient generality to provide a framework for the analysis of the proposed strategy and associated closed-loop system.

Consider the initial-value problem

$$\dot{x}(t) = F(t, x(t), (\widehat{T}x)(t)), \quad x(t) \in \mathcal{D}, \quad x|_{[-h,0]} = x^0 \in C([-h,0]; \mathbb{R}^N), \quad x^0(0) \in \mathcal{D}, \tag{18}$$

where $\mathcal{D} \subset \mathbb{R}^N$ is a non-empty open set, $\widehat{T}$ is a causal operator of class $\mathcal{T}$ and $F : [-h, \infty) \times \mathcal{D} \times \mathbb{R}^K \to \mathbb{R}^N$ is a Carathéodory function*. By a solution of (18) on $[-h, \omega)$, we mean a

*A function $\theta : [-h, \infty) \times \mathcal{D} \times \mathbb{R}^K \to \mathbb{R}^N$, where $\mathcal{D} \subset \mathbb{R}^N$ denotes a non-empty open set, is deemed to be a *Carathéodory function* if, and only if, (i) $\theta(t, \cdot, \cdot)$ is continuous for almost all $t \in \mathbb{R}$, (ii) $\theta(\cdot, x, w)$ is measurable for each fixed $(x, w) \in \mathcal{D} \times \mathbb{R}^K$, and (iii) for each compact $\mathcal{C} \subset \mathcal{D} \times \mathbb{R}^K$ there exists $\kappa \in L^1_{\mathrm{loc}}([-h, \infty); \mathbb{R}_{\geq 0})$ such that $\|\theta(t, x, w)\| \leq \kappa(t)$ for almost all $t \in [-h, \infty)$ and all $(x, w) \in \mathcal{C}$.

function $x \in C([-h,\omega);\mathbb{R}^N)$, with $\omega \in (0,\infty]$ and $x|_{[-h,0]} = x^0$, such that $x|_{[0,\omega)}$ is absolutely continuous and satisfies the differential equation in (18) for almost all $t \in [0,\omega)$, and $x(t) \in \mathcal{D}$ for all $t \in [0,\omega)$; x is maximal if it has no right extension that is also a solution. We remark that a solution $x : [-h,\omega) \to \mathbb{R}^N$ is required to take its values $x(t)$ in the prescribed set $\mathcal{D}$ only for $t \in [0,\omega)$; the values $x^0(t)$ of the continuous initial function are unconstrained for $t \in [-h,0)$.

Theorem 5 *Let $\mathcal{D} \subset \mathbb{R}^N$ be non-empty and open, let $\widehat{T}$ be a class $\mathcal{T}$ operator and $x^0 \in C([-h,0];\mathbb{R}^N)$ with $x^0(0) \in \mathcal{D}$. Assume $F : [-h,\infty) \times \mathcal{D} \times \mathbb{R}^K \to \mathbb{R}^N$ is a Carathéodory function. There exists a solution $x : [-h,\omega) \to \mathbb{R}^N$, $x([0,\omega)) \subset \mathcal{D}$, of the initial-value problem (18) and every solution can be extended to a maximal solution; moreover, if F is locally essentially bounded and $x : [-h,\omega) \to \mathbb{R}^N$, $x([0,\omega)) \subset \mathcal{D}$, is a maximal solution with $\omega < 0$, then, for every compact set $\mathcal{C} \subset \mathcal{D}$, there exists $t' \in [0,\omega)$ such that $x(t') \notin \mathcal{C}$.*

Proof. Since $\mathcal{D}$ is open and $x^0(0) \in \mathcal{D}$, the existence of $\omega > 0$ and a solution $x : [-h,\omega) \to \mathbb{R}^N$, with $x([0,\omega)) \subset \mathcal{D}$, is a consequence of [2, Theorem 3]. That every solution can be maximally extended may be concluded via the following argument (a modification of that used in the proof of [2, Theorem 3]). Let $x : [-h,\omega) \to \mathbb{R}^N$, $x([0,\omega)) \subset \mathcal{D}$, be a solution of (18). Define

$$\mathcal{A} := \left\{(\rho,\xi) \mid \omega \le \rho \le \infty,\ \xi : [-h,\rho) \to \mathbb{R}^N,\ x([0,\rho)) \subset \mathcal{D},\ \text{is a solution of (18)},\quad \xi|_{[-h,\omega)} = x\right\},$$

that is, the set comprising the solution x and all proper right extensions of x that are also solutions. On this non-empty set define a partial order $\preceq$ by

$$(\rho_1,\xi_1) \preceq (\rho_2,\xi_2) \quad \Longleftrightarrow \quad \rho_1 \le \rho_2 \text{ and } \xi_1(t) = \xi_2(t) \text{ for all } t \in [-h,\rho_1).$$

Let $\mathcal{O}$ be a totally ordered subset of $\mathcal{A}$. Let $P := \sup\{\rho \mid (\rho,\xi) \in \mathcal{O}\}$ and let $\Xi : [-h,P) \to \mathbb{R}^N$ be defined by the property that, for every $(\rho,\xi) \in \mathcal{O}$, $\Xi|_{[0,\rho)} = \xi$. Then (P,Ξ) is in $\mathcal{A}$ and is an upper bound for $\mathcal{O}$. By Zorn's Lemma, it follows that $\mathcal{A}$ contains at least one maximal element.

Now let $F : [-h,\infty) \times \mathcal{D} \times \mathbb{R}^K \to \mathbb{R}^N$ be locally essentially bounded. Assume that $x \in C([-h,\omega);\mathbb{R}^N)$, $x([0,\omega)) \subset \mathcal{D}$, is a maximal solution of (18) with $\omega < \infty$. Seeking a contradiction, suppose that there exists a compact subset $\mathcal{C}$ of the set $\mathcal{D}$ with $x([0,\omega)) \subset \mathcal{C}$. By compactness of $\mathcal{C}$ and local essential boundedness of F, together with Property 2 (Definition 1) of $\widehat{T}$, it follows that $\dot{x}$ is essentially bounded. Therefore, x is uniformly continuous and so extends to a continuous function $\tilde{x} : [-h,\omega] \to \mathbb{R}^N$ with $\tilde{x}(\omega) \in \mathcal{C} \subset \mathcal{D}$. Consider the initial-value problem

$$\left.\begin{aligned} &\dot{v}(t) = F\big(t+\omega, v(t), (S_\omega \widehat{T} S_{-\omega} v)(t)\big), && v(t) \in \mathcal{D}, \\ &v|_{[-(h+\omega),0]} = v^0 := S_\omega \tilde{x} \in C([-(h+\omega),0];\mathbb{R}^N), && v^0(0) \in \mathcal{D}\,, \end{aligned}\right\} \tag{19}$$

which can be identified as an initial-value problem of the form (18), with h, F and $\widehat{T} \in \mathcal{T}$ replaced by $\tilde{h} = h+\omega$, $\tilde{F} : (t,v,w) \mapsto F(t-h,v,w)$ and $\widetilde{T} = S_\omega \widehat{T} S_{-\omega} \in \mathcal{T}$ (recall (3)), respectively. Therefore, the above existence result applies to conclude that, for some $\tilde{\omega} > 0$, the initial-value problem (19) has a solution $v : [-(h+\omega),\tilde{\omega}) \to \mathbb{R}^N$, $v([0,\tilde{\omega})) \subset \mathcal{D}$. Define $x^e : [-h,\omega+\tilde{\omega}) \to \mathbb{R}^N$ by $x^e(t) := v(t-\omega) = (S_{-\omega}v)(t)$. Then, $x^e([0,\omega+\tilde{\omega})) \subset \mathcal{D}$ and

$$\dot{x}^e(t) = \dot{v}(t-\omega) = F(t, v(t-\omega), (\widetilde{T}v)(t-\omega)) = F(t, x^e(t), (\widehat{T}x^e)(t)) \quad \text{for a.a. } t \in [0,\omega+\tilde{\omega}).$$

Therefore, $x^e : [-h,\omega+\tilde{\omega}) \to \mathbb{R}^N$, $x([0,\omega+\tilde{\omega})) \subset \mathcal{D}$, is a solution of (18) and is a proper right extension of the solution x, contradicting maximality of the latter. This completes the proof. □

6 Control performance

Before analysing dynamic performance under the proposed control, we highlight a fundamental property of the system class $\mathcal{S}$. This property informs the intuition behind the proposed control.

6.1 A high-gain property of the system class $\mathcal{S}$

Proposition 6 *Let $\varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0};\mathbb{R}_{\geq 0})$ be non-decreasing with $\varphi(t) > 0$ for all $t \in \mathbb{R}_{\geq 0}$. For each $(p, f, T) \in \mathcal{S}$ and $r \in W^{1,\infty}(\mathbb{R}_{\geq 0};\mathbb{R}^M)$, there exists $k^* > 0$ such that, for all $k \geq k^*$ and all initial data $y^0 \in C([-h, 0];\mathbb{R}^M)$ with $\varphi(0)\,\|y^0(0) - r(0)\| < 1$, the control*

$$u(t) = -k\,[y(t) - r(t)] \tag{20}$$

applied to system (1) yields the closed-loop, initial-value problem

$$\dot y(t) = f\big(p(t), (Ty)(t), -k[y(t) - r(t)]\big), \quad y^0 \in C([-h, 0];\mathbb{R}^M), \quad \varphi(0)\|y^0(0) - r(0)\| < 1, \tag{21}$$

which has a solution, every solution y has a maximal extension with interval of existence $[-h,\infty)$, and

$$\varphi(t)\,\|y(t) - r(t)\| \;\leq\; 1 \quad \textit{for all } t \geq 0. \tag{22}$$

Proof. Set $\Lambda := 1/\varphi(0)$, by essential boundedness of r, together with Properties 2 and 3a of Definition 1 of $T \in \mathcal{T}$, $\Delta > 0$ may be chosen so that, for all $z \in C([-h,\infty);\mathbb{R}^M)$ and all $t > 0$,

$$\|z(s) - r(s)\| \leq \Lambda \quad \forall\ s \in [0, t] \quad \Longrightarrow \quad \|(Tz)(s)\| \leq \Delta \quad \text{for almost all } s \in [0, t]. \tag{23}$$

Let compact $\mathcal{P} \subset \mathbb{R}^P$ be such that $p(t) \in \mathcal{P}$ for almost all $t \in \mathbb{R}_{\geq 0}$, write $\lambda := \lim_{t\to\infty}(1/\varphi(t)) = 1/\|\varphi\|_\infty > 0$, and define the compact annulus $\mathcal{A}$ and the compact set $\mathcal{C}$ as follows

$$\mathcal{A} := \{e \in \mathbb{R}^M |\ \lambda/2 \leq \|e\| \leq \Lambda\}, \qquad\qquad \mathcal{C} := \mathcal{P} \times \overline{\mathbb{B}}_\Delta \subset \mathbb{R}^P \times \mathbb{R}^Q\,.$$

Let $\gamma_{\mathcal{C}}$ be defined as in (5) and so, in view of Remark 4(ii), it has the property that $\gamma_{\mathcal{C}}(s) \to \infty$ as $s \to \infty$. Therefore, we may choose $k^* > 0$ sufficiently large so that

$$\gamma_{\mathcal{C}}(k\|e\|) \;\geq\; 1 + \|\dot r\|_\infty + \|\dot\varphi\|_\infty \Lambda^2 \qquad \text{for all } k \geq k^* \text{ and all } e \in \mathcal{A}\,. \tag{24}$$

Fix $k \geq k^*$ and let $y^0 \in C([-h, 0];\mathbb{R}^M)$ be such that $\varphi(0)\,\|y^0(0) - r(0)\| < 1$. Let $\mathcal{D} = \mathbb{R}^M$. By Theorem 5, the initial-value problem (21) has a solution and every solution has a maximal extension. Let $y : [h, \omega) \to \mathbb{R}^M$, $\omega > 0$, be a maximal solution. Write $e(t) = y(t) - r(t)$ for all $t \in [-h, \omega)$ and define

$$I := \{t \in [0, \omega)|\ (e(t), p(t), (Ty)(t)) \in \mathcal{A} \times \mathcal{C}\}\,.$$

Then, in view of (24), we have

$$\begin{aligned}\frac{d}{dt}\,\|e(t)\|^2 \;&=\; 2\,\langle e(t), f(p(t), (Ty)(t), k\|e(t)\|) - \dot r(t)\rangle \\ &\leq\; -2\left[\gamma_{\mathcal{C}}(k\|e(t)\|) - \|\dot r\|_\infty\right]\|e(t)\| \;\leq\; -\,\lambda \qquad \text{for almost all } t \in I.\end{aligned} \tag{25}$$

Therefore, again invoking (24) and recalling that $\varphi(t)\|e(t)\| < 1$ for all $t \in [0, \omega)$, we have

$$\begin{aligned}\frac{d}{dt}\left[\varphi(t)\|e(t)\|\right]^2 =\; &\varphi(t)^2\left[\frac{d}{dt}\,\|e(t)\|^2 + \frac{2\dot\varphi(t)}{\varphi(t)}\,\|e(t)\|^2\right] \\ \leq\; &-2\,\varphi(t)^2\,\left[\gamma_{\mathcal{C}}(k\|e(t)\|) - \|\dot r\|_\infty - \|\dot\varphi\|_\infty\Lambda^2\right]\|e(t)\| \;\leq\; -2\,\varphi(t)^2\,\lambda \\ <\; &0 \qquad\qquad \text{for almost all } t \in I.\end{aligned} \tag{26}$$

Next, we claim that $\|e(t)\| < \Lambda$ for all $t \in [0, \omega)$. Suppose that the claim is false. Then, by continuity and since $e(0) = y^0(0) - r(0) \in \mathcal{F}_\varphi(0)$, $\hat{t} := \min\{t \in [0,\omega)|\ \|e(t)\| = \Lambda\} > 0$ is a well-defined positive number. Moreover, since $\lambda \le \Lambda$, there exists $\hat{s} \in [0, \hat{t}\,)$ such that $\|e(t)\| > \lambda/2$ for all $t \in [\hat{s}, \hat{t}\,]$. Clearly, $[\hat{s}, \hat{t}\,] \subset I$ and so, invoking (26), we arrive at a contradiction:

$$0 < \varphi(\hat{t}\,)\lambda < \varphi(\hat{t}\,)\,[2\Lambda - \lambda] < \varphi(\hat{t}\,)\,\big[2\|e(\hat{t}\,)\| - 2\|e(\hat{s})\|\big] < 2\varphi(\hat{t}\,)\|e(\hat{t}\,)\| - 2\varphi(\hat{s})\|e(\hat{s})\| < 0.$$

Therefore $\|e(t)\| < \Lambda$ for all $t \in [0, \omega)$ and so, by boundedness of r, y is bounded. Therefore, by Theorem 5, $\omega = \infty$.

Now, we show that there exists $s \ge 0$ such that $\|e(s)\| < \lambda/2$. Seeking a contradiction, suppose otherwise. Then $\lambda/2 \le \|e(t)\| < \Lambda$ for all $t \in \mathbb{R}_{\ge 0}$ and so $\mathbb{R}_{\ge 0} \subset I$ which, together with (25) yields the contradiction: $\|e(t)\|^2 \le \|e(0)\|^2 - \lambda\, t$ for all $t \ge 0$.

Our next step is to show that

$$\|e(t)\| < \lambda \qquad \text{for all} \quad t \ge s^* := \min\{s \ge 0|\ \|e(s)\| \le \lambda/2\}.$$

Note that s^* is well defined since e is continuous and $\|e(s)\| < \lambda/2$ for some $s > 0$. Again seeking a contradiction, suppose $\|e(t)\| \ge \lambda$ for some $t > s^*$. Define

$$t^* := \min\{t \ge s^*|\ \|e(t)\| = \lambda\} \quad \text{and} \quad t_* := \max\{t \in [s^*, t^*]|\ \|e(t)\| = \lambda/2\}.$$

Then $[t_*, t^*] \subset I$ which, together with (25), leads to the contradiction: $\lambda/2 < \lambda = \|e(t^*)\| \le \|e(t_*)\| = \lambda/2$.

We now have $\varphi(t)\|e(t)\| \le \lambda^{-1}\|e(t)\| < 1$ for all $t \ge s^*$. It remains to prove that $\varphi(t)\|e(t)\| < 1$ for all $t \in [0, s^*]$. If $s^* = 0$ then the result is immediate. Assume $s^* > 0$. Then $[0, s^*] \subset I$ and so, by (26), $\varphi(t)\|e(t)\| \le \varphi(0)\|e(0)\| < 1$ for all $t \in [0, s^*]$. This completes the proof. □

Proposition 6 implies that, if $\mathcal{F}_\varphi$ is any prescribed performance funnel of the admissible class and $\varphi(0) > 0$, then for each admissible system $(p, f, T) \in \mathcal{S}$ and reference signal r of class $W^{1,\infty}$, there exists a threshold gain value k^* such that for each fixed $k \ge k^*$ and all initial data with $\varphi(0)\|y^0(0) - r(0)\| < 1$, the control $u(t) = -k[y(t) - r(t)]$ ensures that the tracking error evolves within the performance funnel $\mathcal{F}_\varphi$. Evidently, the threshold value k^* depends on the plant data (p, f, T) and on the reference signal r and so is of limited use as the basis of a control design. Proposition 6 does, however, serve to highlight the inherent stability property of the system class under high-gain feedback.

6.2 Tracking within a prescribed performance funnel

In the ensuing Theorem 7, the main result of the paper, the high-gain property underpins the proposed controller structure which ensures (a) prescribed funnel performance for every admissible system (p, f, T) and reference signal r, and (b) boundedness of the control and of the attendant gain function.

Theorem 7 *Let $\alpha : [0, 1) \to \mathbb{R}_{\ge 0}$ be continuous, strictly increasing and unbounded. Let $\varphi \in W^{1,\infty}(\mathbb{R}_{\ge 0}; \mathbb{R}_{\ge 0})$ with $\varphi(s) > 0$ for all $s > 0$ and $\liminf_{s\to\infty} \varphi(s) > 0$. For every system $(p, f, T) \in \mathcal{S}$, every reference signal $r \in W^{1,\infty}(\mathbb{R}_{\ge 0}; \mathbb{R}^M)$, and all initial data $y^0 \in C([-h, 0]; \mathbb{R}^M)$ with $\varphi(0)\,\|y^0(0) - r(0)\| < 1$ the control*

$$u(t) = -k(t)\big[y(t) - r(t)\big], \quad k(t) = \alpha\big(\varphi(t)\|y(t) - r(t)\|\big) \tag{27}$$

applied to system (1) yields the closed-loop initial-value problem

$$\left.\begin{aligned} &\dot y(t) = f\left(p(t), (Ty)(t), -\alpha\big(\varphi(t)\|y(t)-r(t)\|\big)\,[y(t)-r(t)]\right), \\ &y|_{[-h,0]} = y^0 \in C([-h,0];\mathbb{R}^M), \quad \varphi(0)\,\|y^0(0)-r(0)\| < 1\,, \end{aligned}\right\} \tag{28}$$

which has a solution and every solution has a maximal extension.

Every maximal solution $y : [-h,\omega) \to \mathbb{R}^M$ of (28) has the properties:

(i) $\omega = \infty$*;*

(ii) there exists $\varepsilon \in (0,1)$ such that $\varphi(t)\,\|y(t)-r(t)\| \le 1-\varepsilon$ for all $t \ge 0$;

(iii) the continuous functions $u : \mathbb{R}_{\ge 0} \to \mathbb{R}^M$ and $k : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ given by (27) are bounded.

Proof. Let $(p,f,T) \in \mathcal{S}$, $r \in W^{1,\infty}(\mathbb{R}_{\ge 0};\mathbb{R}^M)$ and $y^0 \in C([-h,0];\mathbb{R}^M)$ with $\varphi(0)\,\|y^0(0) - r(0)\| < 1$. Writing $e(t) = y(t) - r(t)$ and introducing the artifact $z(t) = t$, system (28) may be expressed in the form

$$\left.\begin{aligned} &\dot e(t) = f\left(p(t), (T(e+r))(t), -\alpha\big(\varphi(z(t))\|e(t)\|\big)e(t)\right) - \dot r(t), \\ &\dot z(t) = 1, \\ &(e(t),z(t)) \in \mathcal{D} := \big\{(e,z) \in \mathbb{R}^M \times \mathbb{R} \mid \varphi(|z|)\|e\| < 1\big\}, \\ &(e,z)|_{[-h,0]} = (y^0 - r|_{-h,0]}\,,\, 0) = x^0 \in C([-h,0];\mathbb{R}^M\times\mathbb{R}), \quad \varphi(0)\,\|x^0\| < 1\,, \end{aligned}\right\} \tag{29}$$

which, on writing $x(t) = (e(t),z(t))$, can be interpreted as the initial-value problem (18) with $N = M+1$, $K = Q$, the operator $\widehat T$ defined by $(\widehat T x)(t) = (\widehat T(e,z))(t) := (T(e+r))(t)$ and the locally essentially bounded function $F : [-h,\infty) \times \mathcal{D} \times \mathbb{R}^K \to \mathbb{R}^N$ given by

$$(t,x,w) \mapsto F(t,x,w) = F(t,(e,z),w) := (f(p(t), w, -\alpha(\varphi(|z|)\|e\|)e) - \dot r(t)\,,\, 1)\,.$$

Therefore, by Theorem 5, there exists a solution of the initial-value problem (29) and every solution can be maximally extended. Let $(e,z) : [-h,\omega) \to \mathbb{R}^N$ be a maximal solution, $\omega \in (0,\infty]$. Since $(e(t),z(t)) \in \mathcal{D}$ for all $t \in [0,\omega)$, it follows that $\varphi(t)\|e(t)\| < 1$ for all $t \in (0,\omega)$ which, together with continuity of e, implies that e is bounded and so, by boundedness of r, we infer that y is bounded. Since p is essentially bounded and $T \in \mathcal{T}$ satisfies Property 3 of Definition 1, there exists non-empty compact $\mathcal{C} \subset \mathbb{R}^P \times \mathbb{R}^Q$ such that $(p(t),(Ty)(t)) \in \mathcal{C}$ for almost all $t \in [0,\omega)$. Let $\gamma_{\mathcal{C}}$ be defined as in (5) and so, in view of Remark 4(ii), $\gamma_{\mathcal{C}}(s) \to \infty$ as $s \to \infty$. Now,

$$\begin{aligned} \frac{d}{dt}\|e(t)\|^2 &= 2\langle e(t), f(p(t),(Ty)(t),-k(t)e(t)) - \dot r(t)\rangle \\ &\le -2\left[\gamma_{\mathcal{C}}(k(t)\|e(t)\|) - \|\dot r\|_\infty\right]\|e(t)\| \quad \text{for almost all } t \in [0,\omega). \end{aligned} \tag{30}$$

Fix $\delta \in (0,\omega)$. By properties of φ, there exists a constant $c_1 > 1$ such that $c_1^{-1} \le \varphi(t) \le c_1$ for all $t \in [\delta,\omega)$ and $\dot\varphi(t) \le c_1$ for almost all $t \in [\delta,\omega)$. Define $c_2 := \|\dot r\|_\infty + c_1^3$. In view of (30), we may conclude that

$$\frac{d}{dt}\left[\varphi(t)\|e(t)\|\right]^2 = \varphi(t)^2\frac{d}{dt}\|e(t)\|^2 + 2\varphi(t)\dot\varphi(t)\|e(t)\|^2 \tag{31}$$

$$\begin{aligned} &\le -2\,\varphi(t)^2\left[\left[\gamma_{\mathcal{C}}(k(t)\|e(t)\|) - \|\dot r\|_\infty\right] - \frac{\|\dot\varphi(t)\|}{\varphi(t)^2}\right]\|e(t)\| \\ &\le -2\,\varphi(t)^2\left[\gamma_{\mathcal{C}}(k(t)\|e(t)\|) - c_2\right]\|e(t)\| \qquad \text{for almost all } t \in [\delta,\omega)\,. \end{aligned} \tag{32}$$

Next we show that the function $k : [0, \omega) \to \mathbb{R}_{\geq 0}$ is bounded. By continuity, k is bounded on $[0, \delta]$. Seeking a contradiction, suppose that k is unbounded on $[\delta, \omega)$. For each $n \in \mathbb{N}$, define

$$\tau_n := \inf\{t \in [\delta, \omega) |\ k(t) = k(\delta) + n + 1\} \quad \text{and} \quad \sigma_n := \sup\{t \in [\delta, \tau_n) |\ k(t) = k(\delta) + n\}.$$

Choosing $N \in \mathbb{N}$ sufficiently large so that $N + k(\delta) \geq \alpha(0)$ yields, for each $n \geq N$,

$$k(t) \geq n + k(\delta) \quad \text{for all } t \in [\sigma_n, \tau_n] \quad \text{and so} \quad \varphi(t)\|e(t)\| \geq \alpha^{-1}(n + k(\delta)) \quad \text{for all } t \in [\sigma_n, \tau_n]\,,$$

where $\alpha^{-1} : [\alpha(0), \infty) \to [0, 1)$ is the inverse of the bijection $\alpha : [0, 1) \to \mathrm{im}(\alpha)$, whence

$$\|e(t)\| \geq c_1\alpha^{-1}(n + k(\delta)) \geq c_1\alpha^{-1}(k(\delta)) =: c_3 \quad \text{for all } t \in [\sigma_n, \tau_n] \ \text{ and all } n \geq N\,.$$

Therefore, since $\gamma_{\mathcal{C}}(s) \to \infty$ as $s \to \infty$, there exists $n^* \in \mathbb{N}$ such that

$$\gamma_{\mathcal{C}}(k(t)\|e(t)\|) > c_2 \qquad \text{for all } t \in [\sigma_{n^*}, \tau_{n^*}]\,,$$

which, together with (32), yields

$$\frac{d}{dt}\,[\varphi(t)\|e(t)\|]^2 < 0 \qquad \text{for almost all } t \in [\sigma_{n^*}, \tau_{n^*}]\,.$$

Thus, $\varphi(\tau_{n^*})\|e(\tau_{n^*})\| < \varphi(\sigma_{n^*})\|e(\sigma_{n^*})\|$, whence the contradiction:

$$1 = k(\tau_{n^*}) - k(\sigma_{n^*}) = \alpha\left(\varphi(\tau_{n^*})\|e(\tau_{n^*})\|\right) - \alpha\left(\varphi(\sigma_{n^*})\|e(\sigma_{n^*})\|\right) < 0.$$

This establishes boundedness of k from which, together with boundedness of e, we conclude boundedness of the control u. Again by boundedness of k, there exists $\varepsilon > 0$ such that $\varphi(t)\|e(t)\| \leq 1 - \varepsilon$ for all $t \in [0, \omega)$.

To complete the proof, it remains to show that $\omega = \infty$. By boundedness of e, there exists $E > 0$ such that $\|e(t)\| \leq E$ for all $t \in [0, \omega)$. Suppose $\omega < \infty$. Then

$$\mathcal{C} := \{(e, z) \in \mathbb{R}^M \times \mathbb{R}_{\geq 0} |\ \varphi(z)\|e\| \leq 1 - \varepsilon,\ \|e\| \leq E,\ z \in [0, \omega]\}$$

is a compact subset of $\mathcal{D}$ with the property $x(t) = (e(t), z(t)) \in \mathcal{C}$ for all $t \in [0, \omega)$, which contradicts the fact that, by Theorem 5, there exists $t' \in [0, \omega)$ such that $x(t') = (e(t'), z(t')) \notin \mathcal{C}$. Therefore, $\omega = \infty$. $\square$

A hypothesis in Theorem 7 is that $\varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$ and so, in particular, φ is bounded. Clearly, this precludes the use of a performance funnel with radius asymptotic to zero. Therefore, the case of tracking with zero asymptotic error is excluded in the analysis. However, exclusion of the latter case is to be expected. We elaborate this observation in the following remark.

Remark 8 Exact asymptotic tracking cannot be achieved in general by a continuous feedback of the form $u(t) = -k(t)e(t)$ with bounded gain k. To see this, consider the simple case of a scalar linear system with a constant reference signal $r = 1$:

$$\dot{y}(t) = y(t) + u(t), \qquad y(0) = y^0 \in \mathbb{R}. \tag{33}$$

Suppose that k is bounded and such that $u(t) = -k(t)e(t)$ achieves exact asymptotic tracking in the sense that $\lim_{t\to\infty} e(t) = 0$. Then $u(t) \to 0$ as $t \to \infty$ and so there exists $T > 0$ such that $\dot{e}(t) = e(t) + 1 + u(t) > 1/2$ for all $t > T$, contradicting the supposition of exact asymptotic tracking of the constant reference signal $r = 1$. Precisely the same argument shows that exact asymptotic stabilization (tracking of the zero function $r = 0$) cannot be achieved for a scalar affine linear system of the form $\dot{y}(t) = a + y(t) + u(t)$ with $a \neq 0$. $\square$

Therefore, in the context of the general system class $\mathcal{S}$ and reference signal class $\mathcal{R}$, neither exact asymptotic tracking nor exact asymptotic stabilization is achievable by a funnel-type control of the form (27) with bounded gain k. However, in the ensuing subsection it is shown that, if the system class is restricted to the class of linear, minimum-phase, relative-degree-one systems of the form (11) and if the tracking problem is reduced to that of stabilization (in the sense of exact tracking of the zero reference signal $r = 0$), then funnel-type control with bounded gain is achievable.

6.3 Asymptotic stabilization of linear systems

It is well-known (see [1]) that asymptotic stabilization of every member of the class of linear systems (11) can be achieved by an adaptive control of the form $u(t) = -k(t)y(t)$, $\dot{k}(t) = \|y(t)\|^2$, and, moreover, the adapting gain k is bounded. (Here, we use the term "asymptotic stabilization" not in the Lyapunov sense but in the weaker sense of global attractivity of the zero state of the controlled system.) An immediate question arises: is stabilization of all systems of this linear class also achievable by a non-adaptive funnel controller of the form (27), whilst maintaining boundedness of k? Proposition 9 below answers this question affirmatively, provided that the function φ satisfies the following:

$$\left.\begin{array}{l} \text{(a)}\quad \varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0},\ \varphi(0) = 0,\ \varphi \not\equiv 0, \text{ is absolutely continuous and non-decreasing;} \\ \text{there exists } c > 1 \text{ such that:} \\ \text{(b)}\quad \varphi(t) \leq c\,\varphi(t/2) \quad \text{for all } t \in \mathbb{R}_{\geq 0}; \\ \text{(c)}\quad \dot{\varphi}(t) \leq c\,[1 + \varphi(t)] \quad \text{for almost all } t \in \mathbb{R}_{\geq 0}. \end{array}\right\} \tag{34}$$

For example, $t \mapsto \varphi(t) = t$ satisfies (34) with $c = 2$.

Proposition 9 *Let $\alpha : [0,1) \to \mathbb{R}_{\geq 0}$ be continuous, strictly increasing and unbounded, and let $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfy (34).*
Then for each system of the form (11), and all initial data $(y^0, z^0) \in \mathbb{R}^M \times \mathbb{R}^{N-M}$ with $\varphi(0)\|y^0\| < 1$, the control

$$u(t) = -k(t)\,y(t), \qquad k(t) = \alpha\big(\varphi(t)\|y(t)\|\big) \tag{35}$$

applied to the system yields the closed-loop initial-value problem

$$\left.\begin{array}{lll} \dot{y}(t) = A_1\,y(t) + A_2 z(t) - k(t)\,CB\,y(t), & (CB)^T + CB > 0, & y(0) = y^0,\ \varphi(0)y^0 < 1, \\ \dot{z}(t) = A_3 y(t) + A_4 z(t), & \sigma(A_4) \subset \mathbb{C}_-, & z(0) = z^0 \end{array}\right\} \tag{36}$$

which has a solution and every solution has a maximal extension. Every maximal solution $y : [0,\omega) \to \mathbb{R}^M$ of (36) has the properties:

(i) $\omega = \infty$;

(ii) there exists $\varepsilon \in (0,1)$ such that $\varphi(t)\|y(t)\| \leq 1 - \varepsilon$ for all $t \geq 0$;

(iii) the continuous functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^M$ and $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by (35) are bounded.

(iv) If φ is unbounded, then $(y(t), z(t)) \to (0,0)$ as $t \to \infty$.

Proof. Defining $T \in \mathcal{T}$ and $p \in L^\infty(\mathbb{R};\mathbb{R}^M)$ as in (12) and introducing the artifact $\zeta(t) = t$, system (36) may be expressed in the form

$$\left.\begin{aligned} &\dot y(t) = p(t) + (Ty)(t) - k(\zeta(t))\, CB\, y(t), \quad \dot\zeta(t) = 1, \\ &(y(t),\zeta(t)) \in \mathcal{D} := \big\{(y,\zeta) \in \mathbb{R}^M \times \mathbb{R} \mid \varphi(|\zeta|)\|y\| < 1\big\}, \quad (y(0),\zeta(0)) = (y^0,\, 0) \in \mathcal{D}, \end{aligned}\right\} \tag{37}$$

which, on writing $x(t) = (y(t),\zeta(t))$, can be interpreted as the initial-value problem (18) with $N = M+1$, $K = M$, $x^0 = (y^0, 0)$, the operator $\widehat{T}$ defined by $(\widehat{T}x)(t) = (\widehat{T}(y,z))(t) := (Ty)(t)$ and the locally essentially bounded function $F : \mathbb{R}_{\geq 0} \times \mathcal{D} \times \mathbb{R}^K \to \mathbb{R}^N$ given by

$$(t,x,w) \mapsto F(t,x,w) = F(t,(y,\zeta),w) := (p(t) + w - \alpha(\varphi(|\zeta|)\|y\|)y)\,,\, 1)\,.$$

Therefore, by Theorem 5, there exists a solution of the initial-value problem (37) and every solution can be maximally extended. Let $(y,\zeta) : [0,\omega) \to \mathbb{R}^N$ be a maximal solution. Since $(y(t),t) = (y(t),\zeta(t)) \in \mathcal{D}$ for all $t \in [0,\omega)$, it follows that $\varphi(t)\|y(t)\| < 1$ for all $t \in (0,\omega)$ which, together with continuity of y, implies that y is bounded.

By properties (34a) and (34b) of φ, $\varphi(t) > 0$ for all $t > 0$. By monotonicity of φ, $\varphi(t) \leq \varphi(1) \leq \varphi(1)\big[1 + ct^p\big]$ for all $t \in [0,1]$ and all $p > 0$. Let $t \in (1,\infty)$ be arbitrary. Choose $n \in \mathbb{N}$ such that $2^{n-1} \leq t \leq 2^n$ and so $1/2 \leq t/2^n \leq 1$. Then,

$$\varphi(t) = \varphi(2^n(t/2^n)) \leq c^n\varphi(t/2^n) \leq c^n\varphi(1) \leq c\varphi(1)2^{(n-1)\ln c/\ln 2} \leq c\varphi(1)ct^{\ln c/\ln 2}\,.$$

Since $t \in (1,\infty)$ is arbitrary, and writing $p := \ln c/\ln 2$, we may now conclude

$$0 \;<\; \varphi(t) \;\leq\; \varphi(1)\,[1 + ct^p] \quad \text{for all } t > 0. \tag{38}$$

Since A_4 has spectrum in the open left half complex plane, there exist $M,\lambda > 0$ such that

$$\|p(t)\| = \|A_2\exp(A_4t)z^0\| \leq M\,\exp(-\lambda t) \quad \text{for all } t \geq 0, \tag{39}$$

and

$$\begin{aligned} \|(Ty)(t)\| &\leq \|A_1\|\|y(t)\| + \left(\int_0^{t/2} + \int_{t/2}^{t}\right)\exp(-\lambda(t-s))\|y(s)\|\,ds \\ &\leq \|A_1\|\|y(t)\| + \frac{M}{\lambda}\left[\exp(-\lambda\, t/2)\max_{s\in[0\,,\,t/2]}\|y(s)\| + \max_{s\in[t/2\,,\,t]}\|y(s)\|\right] \quad \text{for all } t \in [0,\omega). \end{aligned} \tag{40}$$

Next, we will prove boundedness of k. Seeking a contradiction, suppose that k is unbounded. For each $n \in \mathbb{N}$, define

$$\tau_n := \inf\{t \in [0,\omega) \mid k(t) = k(0) + n + 1\} \quad \text{and} \quad \sigma_n := \sup\{t \in [0,\tau_n) \mid k(t) = k(0) + n\}.$$

Note that, if $N \in \mathbb{N}$ is chosen sufficiently large so that $N + k(\delta) \geq \alpha(0)$, then, for each $n \geq N$, $k(t) \geq n + k(0)$ for all $t \in [\sigma_n,\tau_n]$ and so

$$\begin{aligned} 0 \;<\; c_0 := \alpha^{-1}(1 + k(0)) \;<\; &\alpha^{-1}(n + k(0)) \\ &< \; \varphi(t)\|y(t)\| \;\leq\; 1 \qquad \text{for all } t \in [\sigma_n,\tau_n] \text{ and all } n \geq N\,, \end{aligned} \tag{41}$$

where $\alpha^{-1} : [\alpha(0),\infty) \to [0,1)$ is the inverse of the bijection $\alpha : [0,1) \to \operatorname{im}(\alpha)$. Therefore,

$$\sup_{s\in[t/2\,,t]}\|y(s)\| \;<\; \frac{1}{\varphi(t/2)} \;<\; \frac{\varphi(t)}{c_0\,\varphi(t/2)}\,\|y(t)\| \quad \text{for all } t \in [\sigma_n,\tau_n] \text{ and all } n \geq N.$$

By (40), together with boundedness of y and property (a) of φ, we may infer the existence of $c_1 > 0$ such that

$$\|(Ty)(t)\| \;\le\; c_1 \left[\exp(-\lambda t/2) + \|y(t)\|\right] \quad \text{for all } t \in [\sigma_n, \tau_n] \text{ and all } n \ge N. \tag{42}$$

Since the symmetric part of CB is positive definite, $c_2 := \|[(CB)^T + CB]^{-1}\|^{-1} > 0$ is a well defined positive number. Invoking (39), (41), (42), property (34b) of φ and (38), we may conclude the existence of $c_3 > 0$ such that

$$\begin{aligned}
\frac{d}{dt}\left[\varphi(t)\|y(t)\|\right]^2 =&\; 2\,\varphi(t)\dot{\varphi}(t)\,\|y(t)\|^2 + 2\,\varphi(t)^2\,\langle y(t), p(t) + (Ty)(t) - k(t)\,CB\,y(t)\rangle \\
\le&\; \frac{\dot{\varphi}(t)}{\varphi(t)} + 2\,\varphi(t)^2\|y(t)\|\left[\|p(t)\| + \|(Ty)(t)\|\right] - c_2 k(t)\left[\varphi(t)\|y(t)\|\right]^2 \\
\le&\; \frac{\dot{\varphi}(t)}{\varphi(t)} + 2\,\varphi(t)\left[M\exp(-\lambda t) + c_1\exp(-\lambda t/2)\right] + 2c_1 - c_2 c_0^2 k(t) \\
\le&\; c_3 - c_4 k(t) < c_3 - c_2\, c_0^2\, n \quad \text{for almost all } t \in [\sigma_n, \tau_n] \text{ and all } n \ge N.
\end{aligned} \tag{43}$$

Choose $n^* \ge N$ such that $c_3 - c_2 c_0^2 n^* < 0$, in which case we have

$$\frac{d}{dt}\left[\varphi(t)\|y(t)\|\right]^2 < 0 \quad \text{for almost all } t \in [\sigma_{n^*}, \tau_{n^*}]$$

and so $\varphi(\tau_{n^*})\|y(\tau_{n^*})\| < \varphi(\sigma_{n^*})\|y(\sigma_{n^*})\|$, whence the contradiction:

$$1 = k(\tau_{n^*}) - k(\sigma_{n^*}) = \alpha\left(\varphi(\tau_{n^*})\|y(\tau_{n^*})\|\right) - \alpha\left(\varphi(\sigma_{n^*})\|y(\sigma_{n^*})\|\right) < 0.$$

Therefore, k is bounded.

By boundedness of $t \mapsto k(t) = \alpha(\varphi(t)\|y(t)\|)$, there exists $\varepsilon \in (0,1)$ such that $\varphi(t)\|y(t)\| \le 1-\varepsilon$ for all $t \in [0,\omega)$.

By boundedness of y, there exists $E > 0$ such that $\|y(t)\| \le E$ for all $t \in [0,\omega)$. Suppose $\omega < \infty$. Then

$$\mathcal{C} := \{(y,z) \in \mathbb{R}^M \times \mathbb{R} \mid \varphi(z)\|y\| \le 1-\varepsilon,\ \|y\| \le E,\ z \in [0,\omega]\}$$

is a compact subset of $\mathcal{D}$ with the property $x(t) = (y(t), z(t)) \in \mathcal{C}$ for all $t \in [0,\omega)$, which contradicts the fact that, by Theorem 5, there exists $t' \in [0,\omega)$ such a that $x(t') = (y(t'), z(t')) \notin \mathcal{C}$. Therefore, $\omega = \infty$. This establishes assertions (i)-(iii) of the lemma.

Finally, assume that φ is unbounded. Then $\|y(t)\| < 1/\varphi(t) \to 0$ as $t \to \infty$. Since $\sigma(A_4) \subset \mathbb{C}_-$, the second equation in (36) yields that $z(t) \to 0$ as $t \to \infty$. This completes the proof. ☐

In view of assertion (iv) of the Proposition 9 and choosing φ such that $\varphi(0) = 0$, global asymptotic stabilization of every linear system of the form (11) can be achieved by a control of the form $u(t) = -k(t)y(t)$ which is neither adaptive nor dynamic, and with bounded gain function $t \mapsto k(t) = \alpha(\varphi(t)\|y(t)\|)$. In terms of complexity, this compares favourably with the well-known adaptive and dynamic strategy $u(t) = -k(t)y(t)$, $\dot{k}(t) = \|y(t)\|^2$, which also achieves asymptotic stabilization of every member of the same class and ensures that the adaptive gain function k is bounded[†] Moreover, on one hand, the non-adaptive control has an added benefit: the transient behaviour of the output y can be prescribed through choice of φ. On the other hand, the adaptive strategy is applicable to a wider class of linear systems: the condition $(CB)^T + CB > 0$

[†]Note the trade-off: the adaptive control requires an additional dynamic equation when compared with the non-adaptive control, whilst, to ensure asymptotic stabilization, the latter requires an unbounded function φ; however, we stress that $\varphi(t)\|y(t)\| < 1$ for all $t \ge 0$ and that the non-adaptive gain function $t \mapsto \alpha(\varphi(t)\|y(t)\|)$ is bounded.

on the high-frequency gain can be weakened to the spectrum condition $\sigma(CB) \subset \mathbb{C}_+$ without compromising its stabilization property. The non-adaptive control cannot tolerate the weakened condition: we elaborate this observation in the following remark.

Remark 10 We construct a counterexample which shows that the assertions of Proposition 9 are invalid if the assumption $(CB)^T + CB > 0$ in (10) is relaxed to the spectrum condition $\sigma(CB) \subset \mathbb{C}_+$.

Consider the two-dimensional linear system, parameterized by $\mu \in \mathbb{R}$:

$$\dot{y}(t) = M_\mu u(t), \quad M_\mu := \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}$$

which is of the admissible class $\mathcal{S}$ if, and only if, $M_\mu^T + M_\mu > 0$ (equivalently, $|\mu| < 2$). Note that $\sigma(M_\mu) = \{1\}$ and so the spectrum condition is valid for all $\mu \in \mathbb{R}$. Let α satisfy the hypotheses of Proposition 9 with $\alpha(0) = 1$, choose $\varphi \equiv 1$, and let $\mathbb{B}$ denote the open unit ball centred at $0 \in \mathbb{R}^2$. If $|\mu| < 2$, then, by Proposition 9, the initial-value problem

$$\dot{y}(t) = -\alpha(\|y(t)\|) M_\mu y(t), \quad y(0) = y^0 \in \mathbb{B}\,, \tag{44}$$

has a solution and every solution y has maximal interval of existence $\mathbb{R}_{\geq 0}$. Moreover, $y(t) \in \mathbb{B}$ for all $t \in \mathbb{R}_{\geq 0}$ and $t \mapsto k(t) = \alpha(\|y(t)\|)$ is bounded.

Now consider the case wherein $|\mu| > 2$. Seeking a contradiction, suppose that the assertions of the lemma are also valid in this case. Then, the for every $y^0 \in \mathbb{B}$, every maximal solution y of (44) has interval of existence $\mathbb{R}_{\geq 0}$; moreover $y(t) \in \mathbb{B}$ for all $t \in \mathbb{R}_{\geq 0}$ and $\sup_{t \geq 0} k(t) =: k^* < \infty$. Let $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ be a maximal solution. Define the bijection $K : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $K(t) := \int_0^t k(s)ds$. Observe that $t \leq K(t) \leq k^* t$ for all $t \in \mathbb{R}_{\geq 0}$. Define $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ by the relation $z(K(t)) = y(t)$ for all $t \in \mathbb{R}_{\geq 0}$. Writing $\tau = K(t)$, we have

$$\frac{d}{d\tau}\, z(\tau) = \frac{1}{k(t)} \frac{d}{dt}\, y(t) = -M_\mu y(t) = -M_\mu z(\tau), \quad z(0) = y^0.$$

Therefore, the orbit $y(\mathbb{R}_{\geq 0})$ of the solution y of the nonlinear initial-value problem (44) coincides with the positive semiorbit $O^+ = \{(\exp -M_\mu \tau) y^0 |\ \tau \in \mathbb{R}_{\geq 0}\}$ of the linear initial-value problem

$$\frac{d}{d\tau} z(\tau) = -M_\mu z(\tau), \quad z(0) = y^0.$$

Recalling that $y(t) \in \mathbb{B}$ for all $t \in \mathbb{R}_{\geq 0}$, it follows that $z(\tau) = (\exp -M_\mu \tau) y^0 \in \mathbb{B}$ for all $\tau \in \mathbb{R}_{\geq 0}$. Since y^0 is an arbitrary point of $\mathbb{B}$, we may infer that $\mathbb{B}$ is positively invariant under the linear flow, that is, $(\exp -M_\mu \tau)(\mathbb{B}) \subset \mathbb{B}$ for all $\tau \in \mathbb{R}_{\geq 0}$. Now, $\langle z, -M_\mu z \rangle = -z_1^2 - z_2^2 - \mu z_1 z_2$ for all $z = (z_1, z_2) \in \mathbb{R}^2$ and since $|\mu| > 2$, there exists $z^* = (z_1^*, z_2^*) \in \mathbb{R}^2$ with $\|z^*\| = 1$ such that $\langle z^*, -M_\mu z^* \rangle =: \varepsilon > 0$. By continuity, there exists $\delta \in (0,1)$ such that $\langle z, -M_\mu z \rangle > \varepsilon/2$ for all $z \in \mathbb{B}$ with $\|z - z^*\| < \delta$. Define $m^* := \|M_\mu\| (= \sqrt{1 + |\mu| + \mu^2})$. Let $y^0 = kz^*$ with $k := 1 - \delta\varepsilon/(2(\varepsilon + 2m^*)) \in (0,1)$. Then $y^0 \in \mathbb{B}$ and so $\|z(\tau)\| = \|(\exp -M_\mu \tau) y^0\| < 1$ for all $\tau \in \mathbb{R}_{\geq 0}$. Then

$$\begin{aligned} \|z(\tau) - z^*\| \ &\leq \|z(\tau) - y^0\| + (1-k) \ \leq \ \int_0^t \|M_\mu z(s)\| ds + (1-k) \leq m^* t + (1-k) \\ &\leq \frac{m^*(1-k^2)}{\varepsilon} + (1-k) \ = \ (1-k)\left(\frac{m^*(1+k)}{\varepsilon} + 1\right) \\ &< \ (1-k)\left(\frac{2m^*}{\varepsilon} + 1\right) \ = \ \frac{\delta}{2} \ < \ \delta \qquad\qquad \text{for all } t \in [0, (1-k^2)/\varepsilon]\,. \end{aligned}$$

Therefore, $(d/d\tau)\|z(\tau)\|^2 = 2\langle z(\tau), -M_\mu z(\tau) \rangle > \varepsilon$ for all $t \in [0, (1-k^2)/\varepsilon]$ and so we arrive at the contradiction: $1 > \|z((1-k^2)/\varepsilon)\|^2 > \|y^0\|^2 + (1-k^2) = 1$.

7 Simulations

The following simulations fulfill a triple purpose: (i) to illustrate the theoretical results of the previous sections, (ii) to compare the "funnel" control with the adaptive control proposed in [2] for a class of systems similar to class $\mathcal{S}$ of the present paper, and (iii) to compare the values of the gain function generated by funnel control with the constant gain value of the feedback controller (20) (the latter being unrealisable in practice as it depends on system data unavailable to the controller). Specifically, we consider a nonlinear system of the form

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} a_1 y_1(t) + a_2 |y_2(t)|^{\frac{1}{2}} + a_3 y_1(t) y_1(t-h_1) + p_1(t) \\ a_4 y_1(t) + a_5 y_2(t)(y_1(t-h_2))^3 + a_6 (\mathcal{B} y_2)(t) + p_2(t) \end{pmatrix} + Bu(t) \tag{45}$$

for constants $a_1, \ldots, a_6 \in \mathbb{R}$, $p = (p_1, p_2) \in L^\infty(\mathbb{R}; \mathbb{R}^2)$ and $B \in \mathbb{R}^{2\times 2}$ with $B^T + B > 0$. Here $\mathcal{B}$ denotes the backlash operator of Figure 4 (with parameter $a > 0$). Let $h := \max\{h_1, h_2\}$, then by defining $y := (y_1, y_2)$, the operator

$$T : C([-h, \infty]; \mathbb{R}^2) \to L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^5), \quad (Ty)(t) = (y_1(t), y_2(t), y_1(t-h_1), y_1(t-h_2), (\mathcal{B}y_2)(t)),$$

$w := (w_1, w_2, w_3, w_4, w_5)$, $p = (p_1, p_2)$, and continuous $f : \mathbb{R}^2 \times \mathbb{R}^5 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\hat{f} : \mathbb{R}^2 \times \mathbb{R}^5 \to \mathbb{R}^2$ by

$$f(p, w, u) := \hat{f}(p, w) + Bu, \qquad \hat{f}(p, w) := \begin{pmatrix} a_1 w_1 + a_2 |w_2|^{\frac{1}{2}} + a_3 w_1 w_3 + p_1 \\ a_4 w_1 + a_5 w_2 w_4^3 + a_6 w_5 + p_2 \end{pmatrix},$$

we recast (45) in the form (1). Let $\mathcal{C} \subset \mathbb{R}^{2\times 5}$ be compact. Then,

$$\min_{(p,w)\in\mathcal{C}} \frac{\langle u, f(p, w, u)\rangle}{\|u\|} \geq - \max_{(p,w)\in\mathcal{C}} \|\hat{f}(p, w)\| + \frac{\langle u, Bu\rangle}{\|u\|} \quad \text{for all } u \in \mathbb{R}^2,$$

and since $B + B^T > 0$ we have Property 3 of Definition 3. Furthermore, $T \in \mathcal{T}$, $p \in L^\infty(\mathbb{R}; \mathbb{R}^2)$, therefore $(p, f, T) \in \mathcal{S}$.

Thus (45) is of class $\mathcal{S}$ for any choice of parameters. For the purposes of numerical simulation, we arbitrarily fix the parameters and initial data of (45) as follows:

$$a_1 = \ldots = a_6 = 1, \; h_1 = 1, \; h_2 = \frac{1}{2}, \; B = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \; (\mathcal{B}y_2)(0) = 0, \; a = \frac{1}{5},$$

where a is the backlash parameter, and assume zero initial data

$$(y_1(t), y_2(t)) = (0, 0) \text{ for all } t \in [-h, 0].$$

Furthermore, as disturbance, we take $p := (p_1, p_2)/5$, where p_1, p_2 are the first two coordinates of the solution of the initial-value problem for the Lorenz system

$$\begin{aligned} \dot{p}_1(t) &= p_2(t) - p_1(t), & p_1(0) &= 1, \\ \dot{p}_2(t) &= 2.8 p_1(t) - 0.1 p_2(t) - p_1(t) p_3(t), & p_2(0) &= 0, \\ \dot{p}_3(t) &= p_1(t) p_2(t) - \tfrac{8}{30} p_3(t), & p_3(0) &= 3. \end{aligned}$$

(The solution is chaotic but bounded on $\mathbb{R}_{\geq 0}$: see, for example, [7, Appendix C].) The reference signal is assumed to be

$$t \mapsto r(t) = (r_1(t), r_2(t)) = (\cos(t), \cos(t/2)).$$

As performance funnel, we select φ as in (9) with parameters

$$\lambda = 1/5, \; \varepsilon = 1/2, \; \tau = 7, \qquad \text{that is,} \quad t \mapsto \varphi(t) = \frac{10\,t}{t+7}. \tag{46}$$

All simulations were performed using `ode45` within MATLAB.

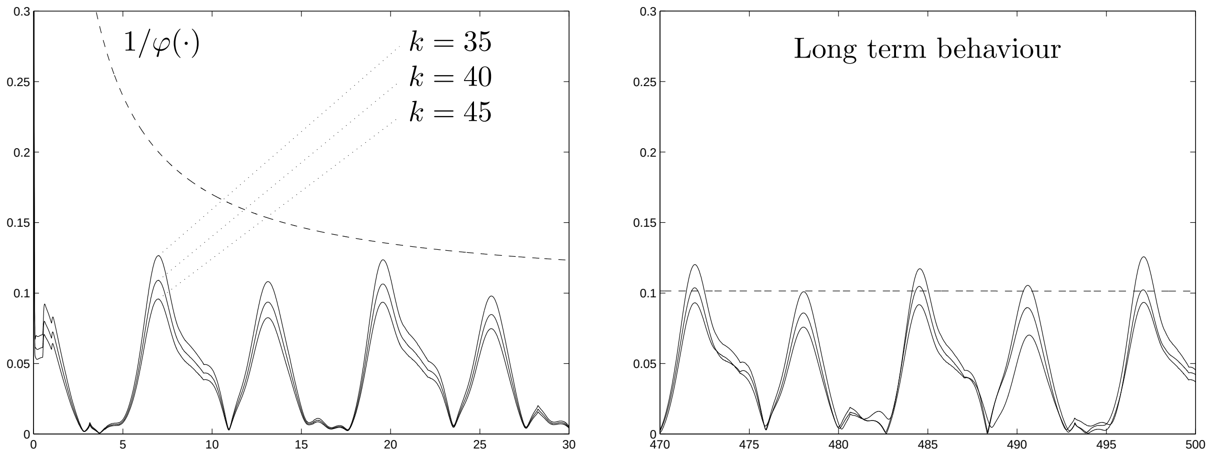


Figure 5: Constant gain feedback control (20) of system (45): $\|e(\cdot)\|$ for $k = 35, 40, 45$; funnel boundary (46).

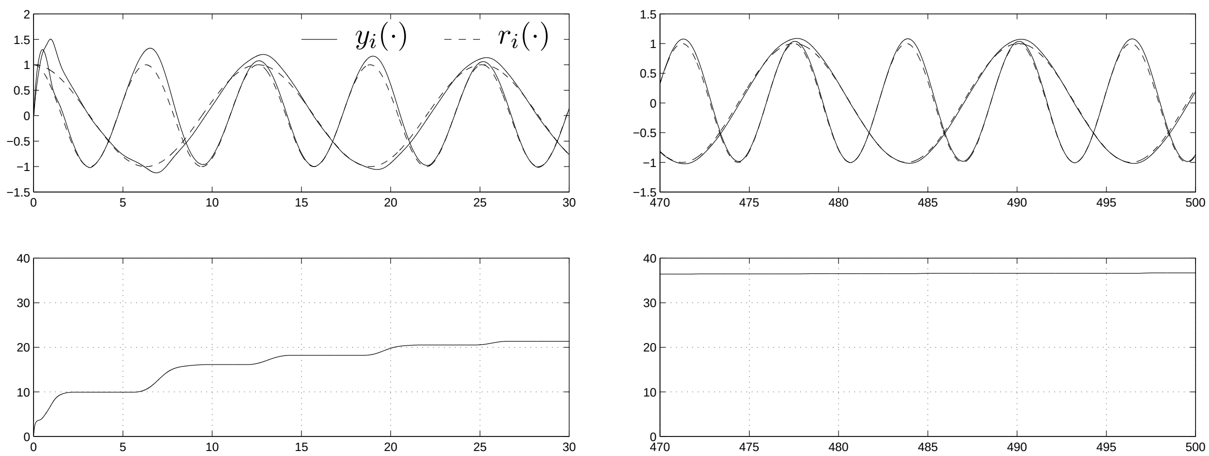


Figure 6: Adaptive λ-tracking control (48) of system (45). Upper: components of system output (solid lines) and reference signal (dashed lines). Lower: adaptive gain $k(\cdot)$.

7.1 Constant gain feedback

Proposition 6 asserts the existence of a k^* such that, for each fixed $k \geq k^*$, the control (20) guarantees that the error evolves within the funnel (however, we reiterate that k^* depends on system data that is unavailable to the controller and so the constant-gain feedback is unrealisable). Figure 5 indicates that, for the chosen initial data and parameters, that a gain value of $k = 35$ is insufficiently large, whilst a value $k = 45$ is more than adequate. Therefore, for the chosen initial data, a threshold gain value of approximately 40 provides a yardstick against which the efficiency of the proposed control (and of the adaptive control of [2]) may be measured. As discussed below, the adaptive control yields a limiting gain value close to the constant threshold value 40 and the funnel control generates a gain function with average value significantly lower than the constant threshold value.

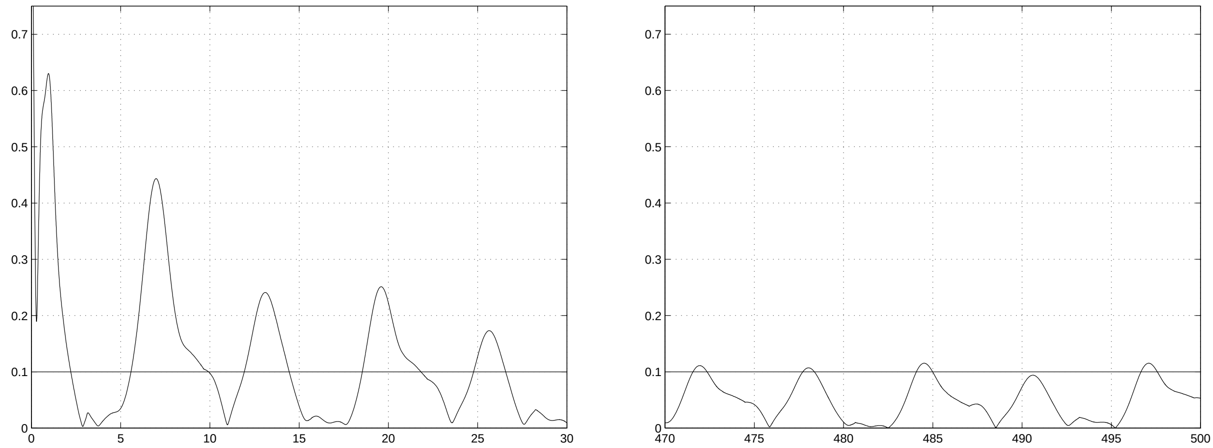


Figure 7: Adaptive λ-tracking control (48) of system (45): error $\|e(\cdot)\|$.

7.2 Adaptive λ-tracking

Reference [2] contains an investigation (related to that of the present paper) of controlled nonlinear functional differential equations of the form

$$\dot{y}(t) = f(p(t), (Ty)(t)) + g(p(t), (Ty)(t), u(t)), \quad y|_{[-h,0]} = y^0 \in C([-h, 0]; \mathbb{R}^M)\,, \tag{47}$$

with essentially bounded p, $T \in \mathcal{T}$ (as in the present paper), and $f : \mathbb{R}^P \times \mathbb{R}^Q \to \mathbb{R}^M$ and $g : \mathbb{R}^Q \times \mathbb{R}^Q \times \mathbb{R}^M \to \mathbb{R}^M$ continuous. In addition, [2] posits:

(a) the existence of a continuous, increasing functions α_f such that, for every compact set $\mathcal{C} \subset \mathbb{R}^P$, there exists c_f with

$$\|f(p, w)\| \le c_f\,[1 + \alpha_f(\|w\|)] \quad \textit{for all} \;\; (p, w) \in \mathcal{C} \times \mathbb{R}^Q\,;$$

(b) the existence of a continuous increasing function α_T and constant c_T such that

$$\|(Ty)(t)\| \le c_T \left[1 + \max_{s\in[0,t]} \alpha_T(\|y(s)\|)\right] \quad \textit{for all} \;\; t \ge 0;$$

(c) for every compact set $\mathcal{C} \subset \mathbb{R}^P$ there exists a positive-definite, symmetric $G \in \mathbb{R}^{M\times M}$ such that

$$\langle Gu, g(p, w, u)\rangle \ge \|u\|^2 \quad \textit{for all } (p, w, u) \in C \times \mathbb{R}^Q \times \mathbb{R}^M\,.$$

Under the above hypotheses, an adaptive control is developed in [2], with a "dead-zone" in the gain adaptation, that achieves the following control objective: for every system of the admissible class, every reference signal $r \in \mathcal{R}$ every prescribed $\lambda > 0$, all variables of the closed-loop system are bounded and the tracking error tends asymptotically to the ball $\mathbb{B}^M_\lambda$, that is, $\lim_{t\to\infty} \operatorname{dist}(\|e(t)\|, [0, \lambda)) = 0$. For the system (45), hypotheses (a), (b) and (c) hold with $\alpha_f : s \mapsto s^4$, $\alpha_T : s \mapsto s$ and $G = I$, in which case the adaptive feedback control of [2] becomes

$$u(t) = -k(t)\,\psi(e(t)), \quad \dot{k}(t) = \max\{0, \|e(t)\| - \lambda\}\,, \quad e(t) := y(t) - r(t)\,, \tag{48}$$

where ψ (continuous) is defined as

$$\psi : \mathbb{R}^M \to \mathbb{R}^M, \quad e \mapsto \psi(e) := \begin{cases} [\|e\| + \alpha_f(\alpha_T(\|e\|))]\,\|e\|^{-1}\,e, & e \ne 0 \\ 0, & e = 0. \end{cases} \tag{49}$$

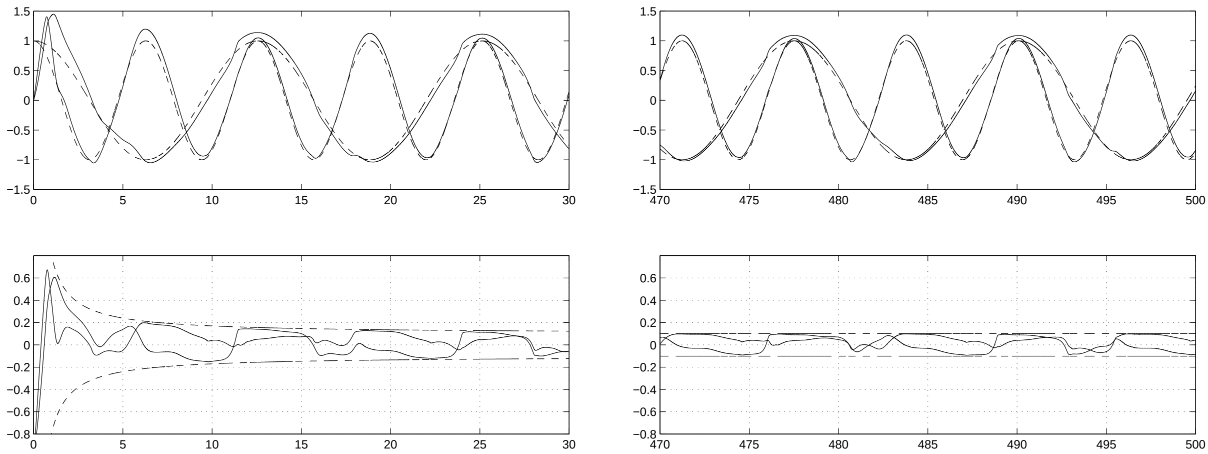


Figure 8: System (45) under funnel control (27) with funnel envelope (46). Upper: components of system output (solid lines) and reference signal (dashed lines). Lower: components of error (solid lines) and funnel boundary (dashed line).

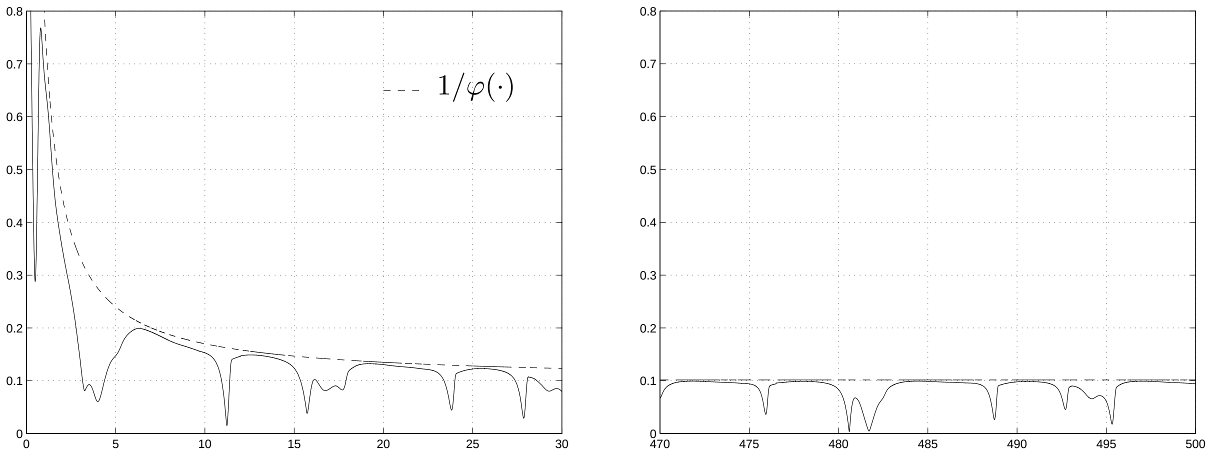


Figure 9: System (45) under funnel control (27) with funnel envelope (46): tracking error $\|e(\cdot)\|$.

For system (45), with the above-chosen parameter values and initial data, performance under the adaptive controller, with initial data $k(t) = 0$ for all $t \in [-h, 0]$ and with $\lambda := \lim_{t\to\infty} 1/\varphi(t) = 1/10$, is shown in Figures 6–7. Convergence of the gain is guaranteed; as indicated by Figure 7, the limiting gain value is close to the constant gain threshold value 40. Note that performance issues, central to the present paper, are not captured in the adaptive approach: the transient behaviour is not predictable and one may not stipulate a time $\tau > 0$ after which the error is guaranteed to evolve in the ball $\mathbb{B}^M_\lambda$.

7.3 Tracking by funnel control

Finally, we consider the feedback control (27) with $\alpha(s) := 1/(1 - s)$ and funnel envelope (46); since $\varphi(0) = 0$ the feasibility condition of (28) is vacuous. For system (45), with the above-chosen parameter values and initial data, Figures 8–10 show the output evolving within the funnel as predicted by Theorem 5.

Notice that the largest "spike" of the gain function k is commensurate with the constant gain threshold value 40, whilst the "average" gain value is considerably lower than this threshold.

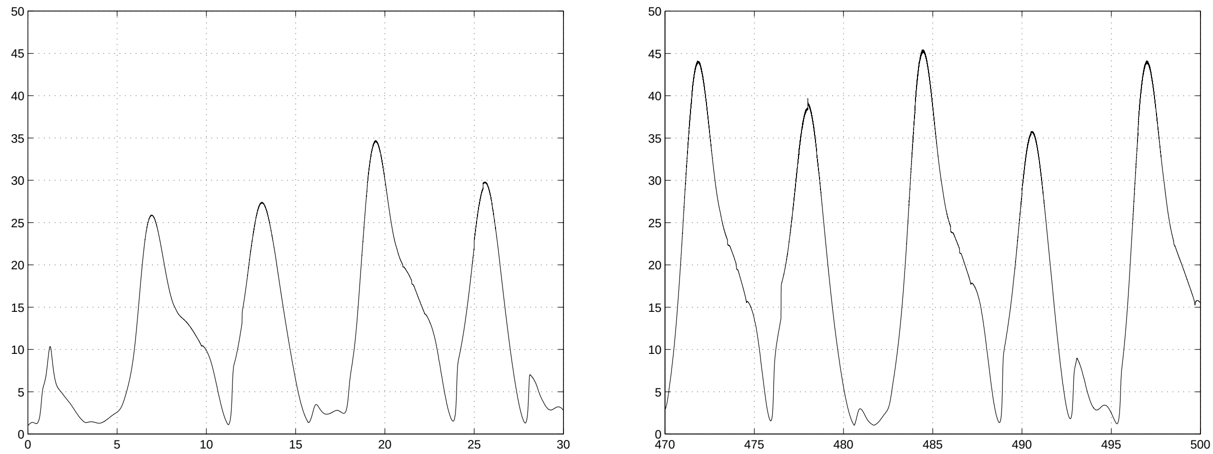


Figure 10: System (45) under funnel control (27) with funnel envelope (46): evolution of gain $k(\cdot)$

We emphasize that, when contrasted with the adaptive control results of Subsection 7.2, "funnel" control ensures, not only asymptotic performance, but also prescribed transient performance; moreover, this performance is achieved without positing hypotheses (a) and (b) required by the adaptive design.

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